Long run growth 2: The Solow model

Why do we see such large differences in level of income across countries and in their growth rate performance? To answer this question we introduce one of the leading models in economics, introduced by Robert Solow (economist at MIT, winner of the Nobel prize in economics in 1987) in the 50s. This is a pretty simple theory but quite powerful in the sense that it can be used to analyze a whole variety of issues. The key point of the theory is that sustained growth and high levels of income per capita in a country can be achieved through saving and capital accumulation. Limits to capital accumulation are posed by the fact that returns to capital are decreasing. Decreasing returns simply means that the benefit of any additional unit of capital are declining with the stock of capital per worker. Suppose I am a weaver and I weave by hands. My output will be quite low. If I purchase a loom I will increase my output a lot. If I buy a second loom I will increase my output but not as much as the first increase because now I have to divide my time between the two looms. If I have 10 looms and I add an 11th loom my output will not really increase by much because I don't have really the time to operate it.

The basic model

For now we will consider the total number of worker in a country constant and equal to L. We will also assume that everybody in this economy work so GDP per capita is also equal to GDP per worker (L is also equal to population).

The theory is built around two simple equations. The first equation is the aggregate production function of a country that is assumed to be of a particular form called Cobb-Douglas

$$Y = AF(K, L) = AK^\alpha L^{1-\alpha}$$

Y here represents output produced, K is the capital stock in place, L is the amount of labor employed and A is a parameter that determines the efficiency of the factors of production. A sometimes is also called "Total Factor Productivity" or TFP. Note the the particular functional form display "Constant Return To Scale" that is a doubling

of both inputs of production (labor and capital) leads to a doubling of output, or more formally

$$AF(kK, kL) = A\left(kK\right)^\alpha \left(kL\right)^{1-\alpha} = kAF(K, L)$$

Another interesting feature of this production function is that it displays constant share of income going to factors of production (under competitive factor markets). Notice that if factor markets are competitive they are paid their marginal product so, for example, wages w are given by

$$w = \frac{\partial AF(K, L)}{\partial L} = (1 - \alpha)\frac{AK^\alpha L^{1-\alpha}}{L}$$

which implies

$$(1 - \alpha) = \frac{Lw}{AK^\alpha L^{1-\alpha}}$$

i.e. the share of income going to labor is constant and equal to $1 - \alpha$. In the data for most countries the share of income going to labor is roughly constant through time, and roughly equal to 70%, hence usually α is set to 0.3.

The second equation follows from the fact that national income is split it between consumption C and investment I

$$Y = C + I$$

For now we are assuming that the economy is closed (no link with the rest of the world) and we do not model the government separately from the private sector. Also for now let's assume that consumers consume a fixed fraction of their income and let's call this fraction $(1 - s)$ so $C = (1 - s)Y$. Using this assumption and the division of national income we obtain

$$\begin{aligned} Y &= C + I = (1 - s)Y + I \\ I &= sY \end{aligned}$$

this implies that investment is a fixed fraction of income. Dividing everything by the total number of workers L and denoting with lower case letters the per-worker variables we get

$$i = sy$$

where

$$\begin{aligned} i &= \frac{I}{L} \\ y &= \frac{Y}{L} \end{aligned}$$

dividing by L both sides of the production function we also get

$$y = AK^\alpha \frac{L^{1-\alpha}}{L} = \left(\frac{K}{L}\right)^\alpha = Ak^\alpha$$

The relation $y = Ak^{\alpha}$ is known as the per-worker production function and it tells how much output per worker is produced given the capital stock per worker. Figure 1 plots the per-worker production function i.e. the output per worker as a function of capital per worker, together with investment per worker as a function of capital per worker. As you can see from the figure the curve is concave, that is its slope (which we call Marginal product of capital or mpk) is decreasing with the level of capital stock per worker. This just reflects decreasing returns of capital with a fixed number of workers. With low levels of capital per worker an increase in capital stock increase output per worker a lot, with high levels of capital per worker increases the same increase in capital stock causes a smaller increase in output.
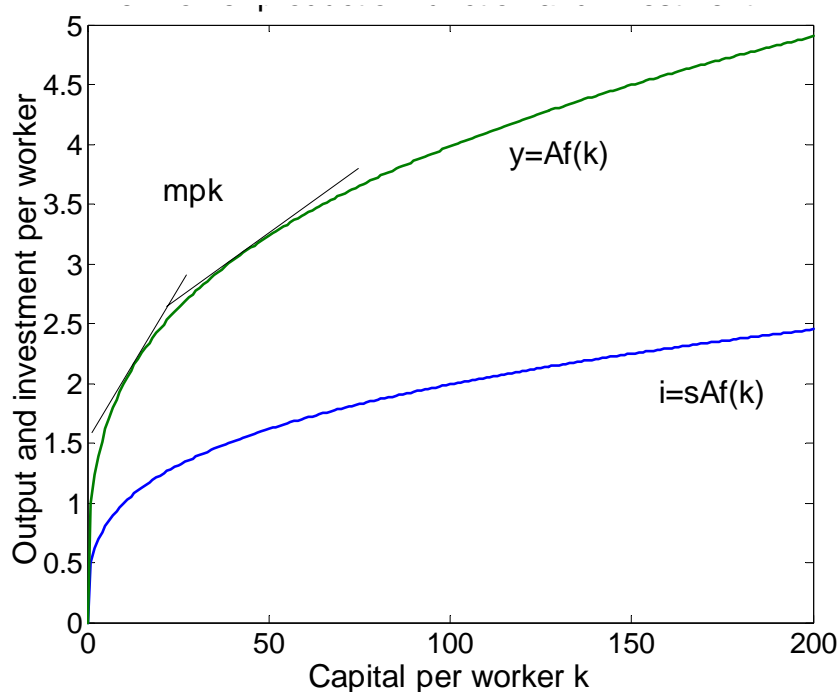

Figure 1: PER WORKER PRODUCTION FUNCTION

How does the capital stock per worker evolve over time?

At each point in time there is investment and thus capital stock per worker goes up. But there is also depreciation and thus the capital stock per worker goes down. Therefore the change in the capital stock per worker (Δk) in each point in time will be given by

$$\Delta k = i - \delta k = sAf(k) - \delta k$$

where δ is the rate of depreciation of physical capital. Figure 2 plots the depreciation rate and the investment as a function of the capital stock per worker. The value of k for which the two lines cross is called the steady state level of capital, that is

the capital for which depreciation is equal to investment. If an economy starts with capital equal to its steady state level it will stay there because $\Delta k = 0$. If an economy starts with a capital below the steady state level then $i > \delta k$ so $\Delta k > 0$ and therefore capital increases. On the other hand if k is above the steady state level $i < \delta k$ so $\Delta k < 0$ and the capital stock will fall toward the steady state level. This model thus predicts that countries will eventually converge toward the steady state level of capital per worker k_{ss}. Clearly this will determine also the level of output and consumption per worker (or per capita) of a given country.
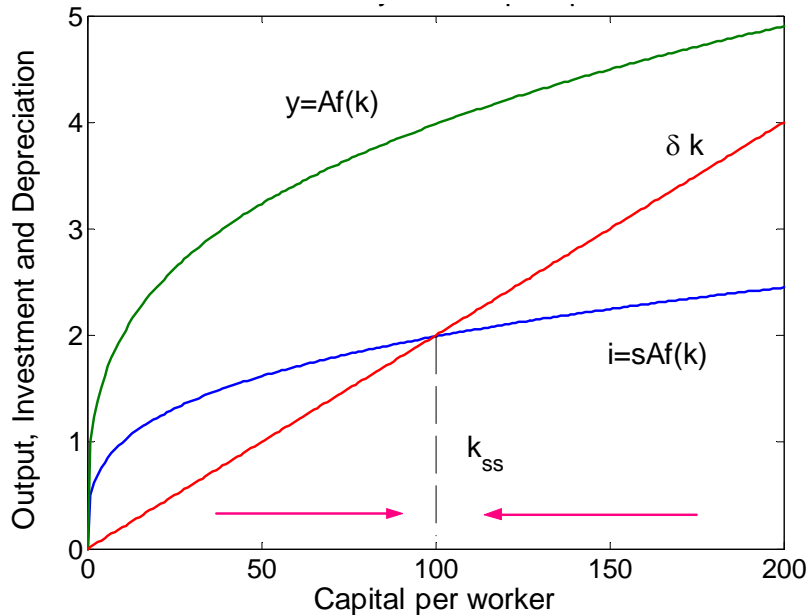

Figure 2: DETERMINATION OF STEADY STATE CAPITAL PER WORKER

So what can we learn about differences in per capita income?

If we believe that all countries have access to the same technology and hence have the same aggregate production function, that depreciation and saving rates are the same then the model predicts that countries will all converge to the same level of income per capita. The model predicts that differences in income per capita are only temporary and are due to differences in the initial stock of capital. In particular the theory predicts that a country that start with a low level of capital stock (and income) per capita should display faster growth and catch up with countries that have high level of income and capital per capita. This is sometimes called the "convergence" hypothesis. This optimistic view is somehow confirmed if we look at a restricted sample of countries (figure 3), at Japan and Germany after WW2 (figure 4) or at the sample of US states (figure 5) where we see that countries or states starting with a lower level of income per capita have displayed a much higher growth rate.
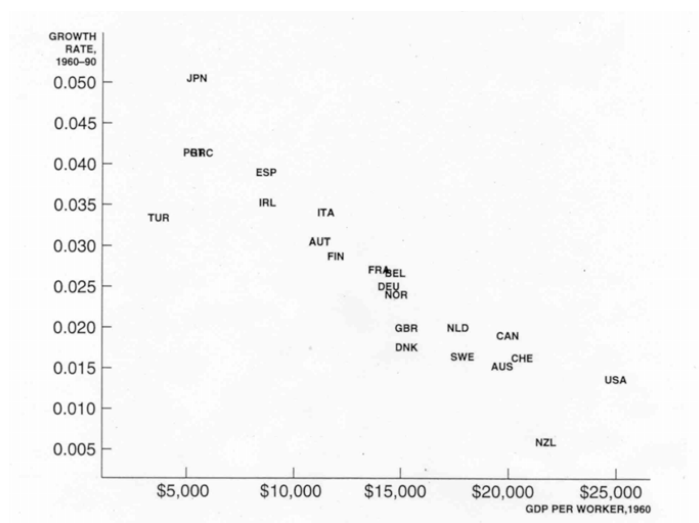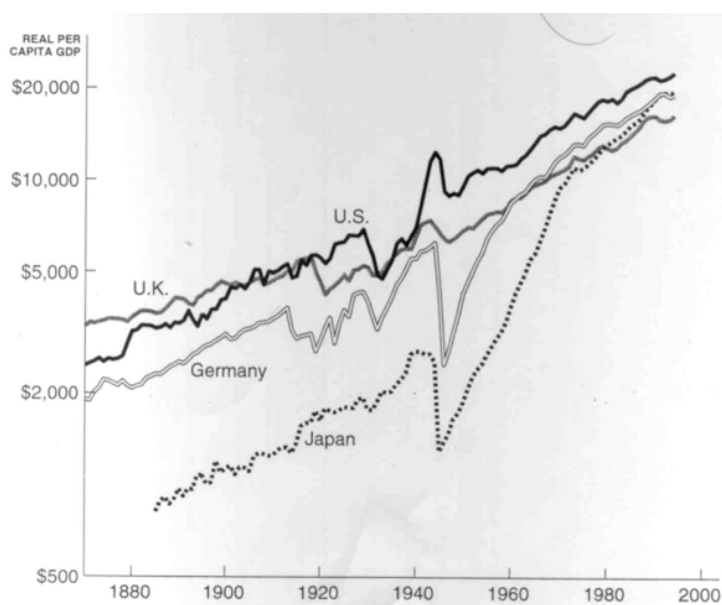
Figure 3: Convergence for OECD countries

Figure 4: Convergence for Japan and Germany after WW2

These pictures though are somehow misleading because includes only industrialized countries or states, that is only countries that have experienced a successful transition to the developed stage. Figure 6 plots the relation between initial income and growth for a larger group of countries and shows that convergence is not present at all and we see countries with a low level of income that have also displayed very low growth, so that in a large cross section of countries we actually observe divergence. This
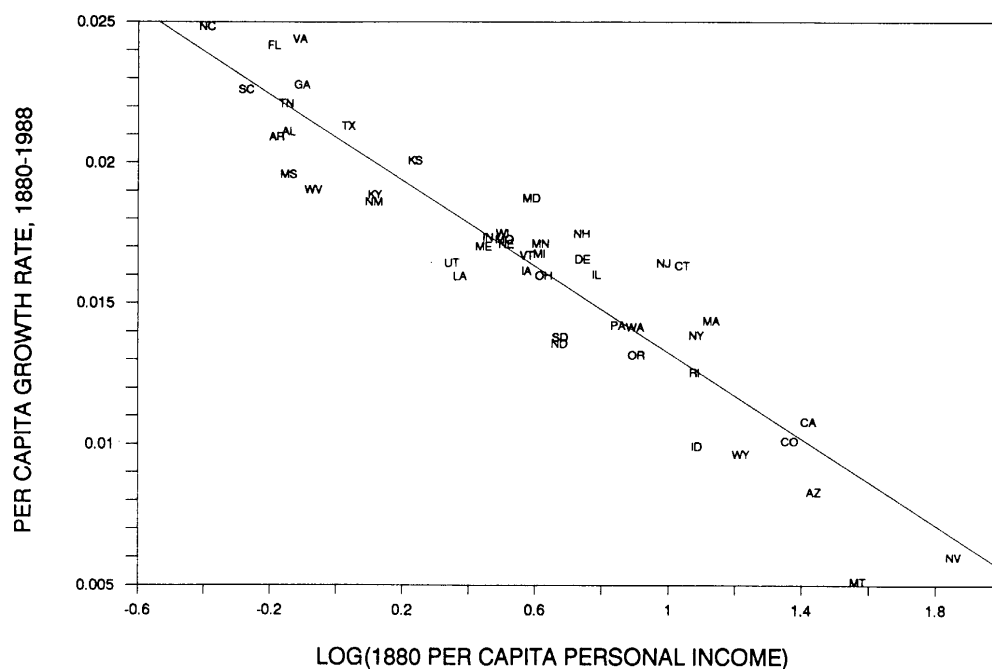
Figure 5: CONVERGENCE FOR US STATES

last picture suggests that either differences in steady states are important or that for some reason countries are not able to converge to their steady states. The conclusion is that there are permanent differences in income per capita that cannot simply be explained by different point along the convergence toward the same steady state. One possibility is then that countries are different and they have different steady states toward which they converge. What are the causes of different steady states? We will now explore various possibilities, such as differences in saving rates or differences in population growth rates.

Differences in saving rates

Consider two countries that are equal in every respect but have different saving rate (see figure 7). The country with the higher saving rates accumulates more capital and therefore converges to an higher level of capital stock and output per worker. This prediction of the model seems roughly consistent with the empirical evidence that emerges from looking at a cross section of countries. Figure 8 suggests that high investment rates are associated with high level of GDP per capita. But does the picture really tells us that if Gambia increases its saving rate from 5% to 10% this will lead to a doubling of GDP per capita in Gambia? We will go back to this point
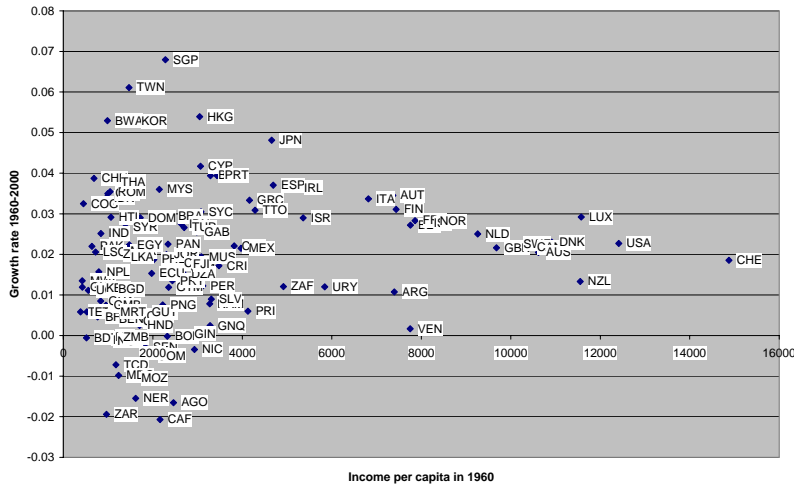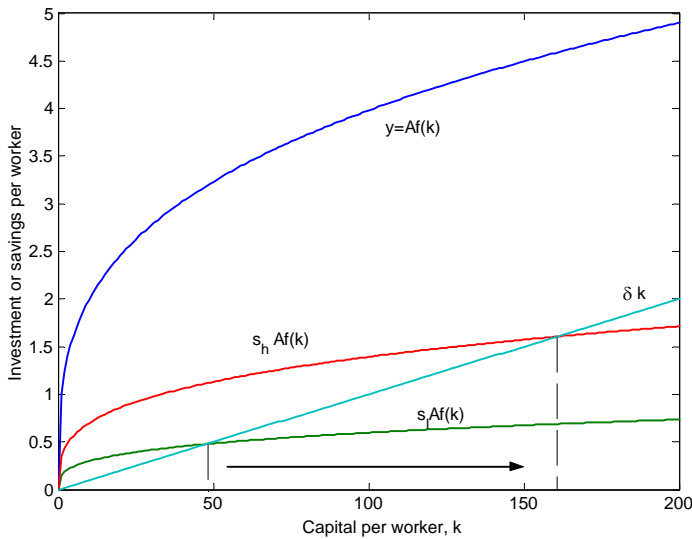
Figure 6: DIVERGENCE

Figure 7: EFFECTS OF DIFFERENCE IN SAVING RATES

later. For now keep in mind that this relation has motivated many policy choices.

The Stalin 5 years plans in the thirties were basically based on this theory and a lot of programs in developing countries in the 60s had as a main goal the increase in saving (even forced saving though taxes) to increase capital stock and achieve a higher output per worker. Even more recently the IMF has stated that "*To achieve gains in real per capita GDP an expansion in private saving and investment is key*".
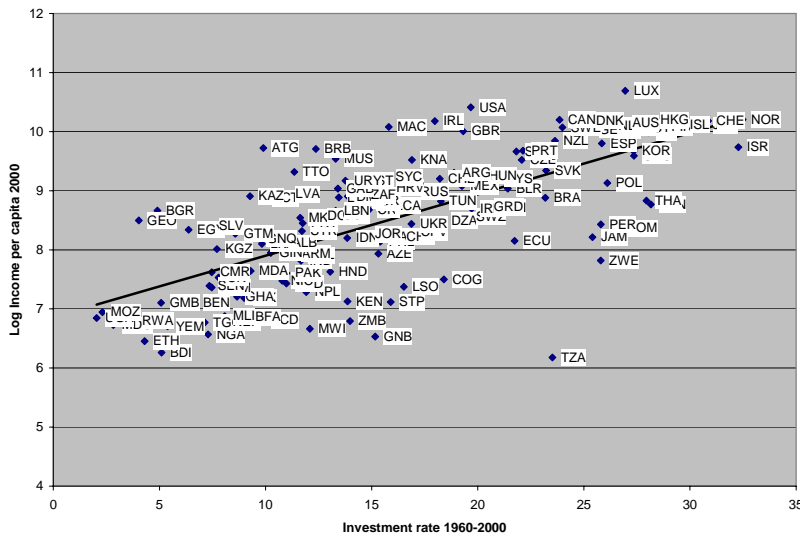
Figure 8: SAVINGS AND INCOME PER CAPITA

The Golden rule of saving

An interesting question is whether higher saving, even if it always yields higher levels of steady state income, always yields higher steady state consumption. The answer is no and the intuition comes from the fact that returns to capital are decreasing. Increasing my saving by 5% always reduces my consumption by 5% but the benefit it will bring in terms of future consumption depends on the returns to capital. If I have little capital returns to capital are high and my consumption will eventually increase. If I have a lot of capital returns to capital are low, the benefits of additional saving are low and consumption might actually decline. The figure shows how the steady state level of consumption depends on the saving rate. Note that it reaches a maximum (the saving line labeled optimal saving) at the point in which the marginal product of capital is equal to the depreciation rate. This saving for which the maximum consumption is achieved is called the "golden rule" of saving.
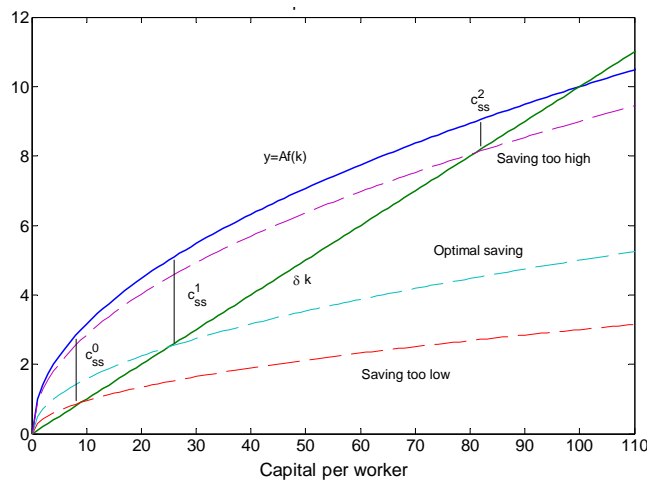
But if savings and investment are so important why do some countries (for example the African countries) fail to realize that and do not save and invest? So far we've been treating savings as an exogenous variable and so we are not really equipped to answer this question. A first simple theory of saving rates might be based on basic needs: some countries do not save simply because they are too poor to save. Suppose that when people have very low income they need all their income to satisfy their basic needs (like food) and thus their saving rate is 0 (They consume all their income). When income reaches a basic level $\bar{c}$ people start saving and save a constant fraction of their income as in the previous example. Also assume that this rate is the same across countries. In this case consumption is given by

$$C = \begin{cases} Y, \text{ if } Y < \bar{c} \\ \bar{c} + (1-s)(Y - \bar{c}) \text{ if } Y > \bar{c} \end{cases}$$

Investment will be given by

$$I = Y - C = C = \begin{cases} 0, \text{ if } Y < \bar{c} \\ s(Y - \bar{c}) \text{ if } Y > \bar{c} \end{cases}$$

the production equation is the same as before but now the Solow diagram can be quite different (see figure 9)

Now the model displays two steady states. Notice that the one steady state (the one with low capital) is unstable in the sense that if capital is above it , it will move toward the higher capital steady state but if a country starts with a capital below that level that country will converge toward a level of 0 capital stock. The situation depicted is called a *poverty trap*. Countries that start with a level of capital stock that is very low will never take off. Notice that here we have reversed the causality: it is not that countries are poor because they don't save but they do not save because they are poor.

The poverty trap model has been the rational for the kind of policy denominated big push, that is a massive investment from abroad that is able to raise the capital stock just above the first steady state so convergence to the higher steady state can happen. The big problem with this theory has been that even though these type of big push policies have been tried repeatedly (from Zambia in the 60s to Cambodia or Lithuania in the 1990s) they do not seem to have generated sustained growth (see figure 10). For example in the period 1960-1988 of 87 developing countries (income per capita below US$ 5000) 47 of them have failed to even improve their standard of living. So why are some countries unable to takeoff even with massive foreign aid?

Differences in Population Growth rates

As Thomas Malthus pointed out long time ago the per capita level of resources is going to depend also on how fast population is growing. In order to incorporate this
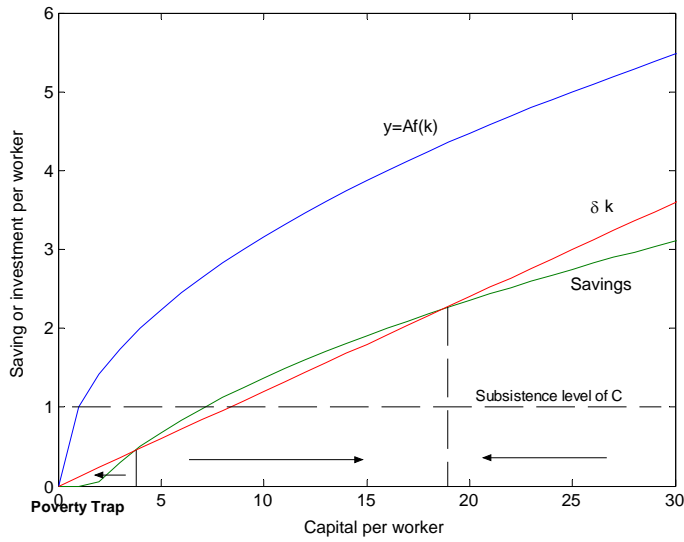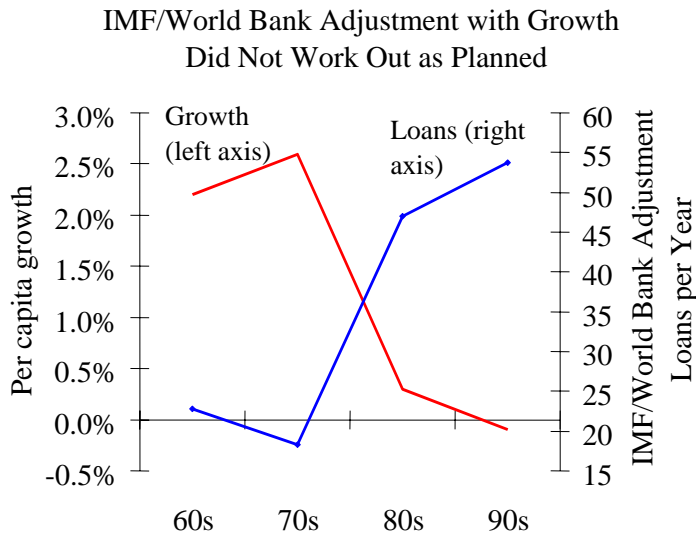
Figure 9: A POVERTY TRAP

Figure 10: EXITING A POVERTY TRAP?

in the basic Solow model let's assume that the population (the number of workers) is growing at a rate of n that is

$$\frac{L_{t+1} - L_t}{L_t} = n$$

Consider again the changes in the per capita capital stock: now they will be given by

$$\Delta k = i - (\delta + n)k$$

Why do we have this extra term n?

Remember that $k = \frac{K}{L}$ and therefore when L is going up k will tend to go down and the higher the rate of population growth the higher the rate of investment is required in order to prevent the per capita capital stock from falling. Consider this example

There is an economy with 10 workers and 10 computers. Computers depreciate at 10% per year. Next year we will have 10 workers and 9 computer. In order to keep the ratio of computer per worker constant we need to buy only one additional computer (1/10 of computer per worker). Suppose now the workers are growing at 20% per year. In this case next year we have 12 workers and 9 computers and in order to keep the ratio constant we need to buy three new computers (3/10 of computer per worker).

Figure 11 plots the Solow diagram modified to include population growth rate and shows that population growth rate shifts the straight line up and thus reduces the steady state level of capital stock per worker (and hence the level of per capita output).
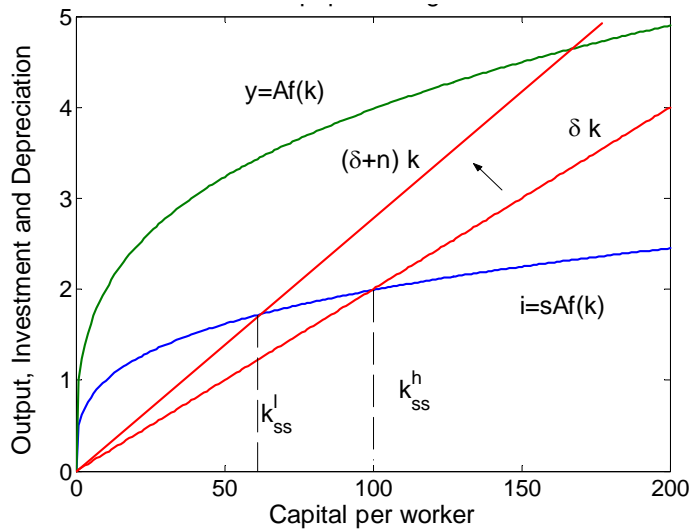

Figure 11: THE SOLOW MODEL WITH POPULATION GROWTH

Notice that there is a basic difference between this case of the Solow model without population growth. In the steady state of the model without growth k is constant and this implies also that K is constant (since L does not grow). Therefore the model predicts no growth in total output or total capital. In the steady state with growth k is constant but now L is growing so the model displays growth in total K and total Y (although no growth in per capita Y). In other words the model predicts, consistently with Malthus, that economies are getting bigger but not richer. If we consider two countries with different population growth another prediction of the model is that the

country with the higher rate of population growth should also have the lower income per capita: this prediction is somehow confirmed by the data (see figure 12)
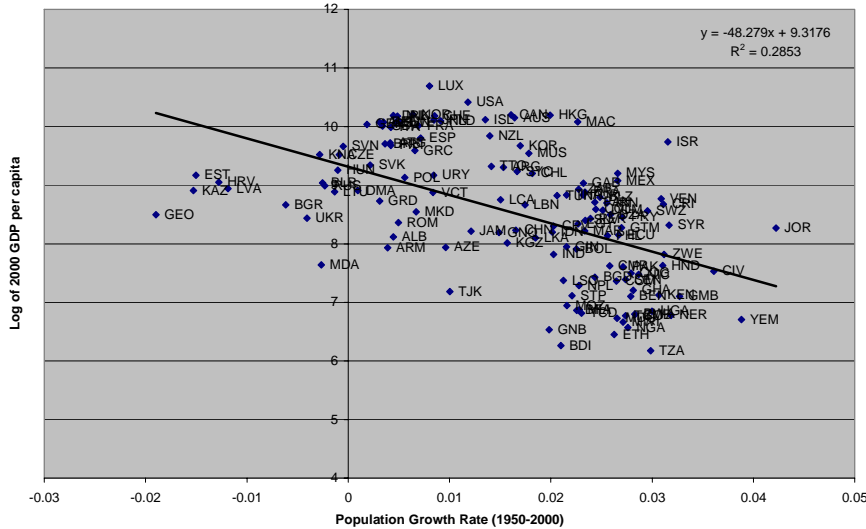

Figure 12: POPULATION GROWTH AND INCOME PER CAPITA

One might think that this theory provides a justification for China to implement a policy of allowing only one child per couple. But again can we then say that is the difference in population growth rates that generate differences in income levels? Even if that was the case, the key question is then why some countries have so high population growth rates if they prove so bad for them? The answer to this question comes by observing figure 13 which shows the typical path of population growth for a country during its development. In class we will discuss how it is reasonable to think that it is not the demographics driving income but rather income (or development) driving the demographic pattern.

The endogeneity problem

The two possible explanations that we considered so far for the large differences in income per capita seem convincing at a first glance (the scatter diagrams reveal association between saving rates or population growth rates and income per capita). But if we think more about the issue we realize that both saving rate and population growth rates are really endogenous variables, meaning that they are themselves influenced by the level of income per capita. That it is not that countries that save more or have fewer kids become rich but it is that countries that become rich have fewer kids and save more. The more recent view of economists that study growth is
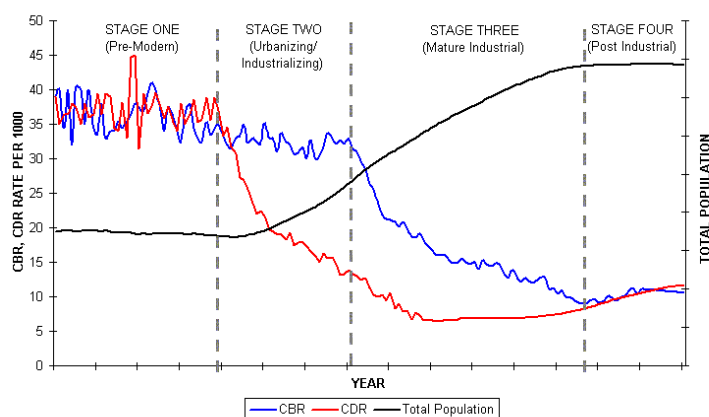
Figure 13: THE DEMOGRAPHIC TRANSITION

that there is a common force that drives at the same time GDP per capita, saving and population growth rates. If this force is not present, programs of forced saving or forced reduction of population growth are doomed to fail. In other words low saving rates or high population rates are like the cough in a person with a flu. It is true that all persons with the flu cough, but forcing the person with the flu not to cough will not make her any better. To really cure the person you need to cure the cause. What is then this force that cause all these countries to be so poor?

Returns to Capital

To understand this force we need to step back and think about an individual consumption saving problem. An average individual in country X has 1 dollar and she needs to decide what to do with it. She can either consume it or invest it and consume the proceeds of the investment next year. People usually prefer consumption today to consumption in the future so we will assume that this person is indifferent between consuming 1 dollar today or $R > 1$ dollars tomorrow. Economists call R the rate of time preference. The higher R the more impatient people are and the more like consuming today rather than investing. The returns of investing in the domestic capital stock are given by MPK (the marginal product of capital that is the slope of the per capita production function in figure 1) plus $1 - \delta$ reflecting the fact that next year part of the investment will be depreciated. Figure 14 plots the return to investment and the rate of time preference of the residents as a function of capital per worker in the country. If the return to investment is higher than the rate of time preference the average individual invests, and investment adds to the capital stock until the return to investment is equal to the rate of time preference. Note that is another way of obtaining the steady state capital stock per worker (and thus income per worker),

but this time saving rate is not exogenous but it is the result of an optimal decision by agents. If the curve representing the returns to investment shifts then the saving behavior will change and will reduce or increase the capital stock per worker. If, on the other hand, the curve reflecting investment incentives does not move, but capital stock increases (following, for example, a massive foreign aid program) the increase will not have long run consequences as residents will simply choose to consume the additional capital stock that is been brought in the country and they will revert to their initial level. In the next lecture we will first provide evidence on how changes in the returns to capital are indeed the engine of growth and then we will study what are the key determinants of these returns.
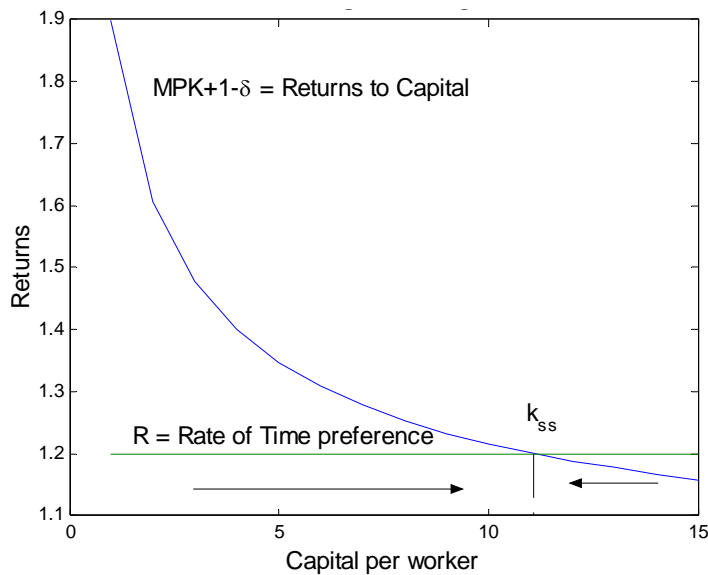

Figure 14: Savings and returns to capital

Concepts you should know

1. Decreasing returns

2. Solow Model

3. Convergence

4. Golden Rule

5. Poverty Trap

6. Solow model with population growth

7. Endogeneity problem

8. Returns to Capital

Review Question

Assume the following production function

$$Y = K^{.5}L^{.5}$$

1) What is the per worker production function $y = f(k)$

2) Assume that the depreciation rate is 10% and that the saving rate is 20% . What is the steady state capital per worker and output per worker.

3) What happens to the output per worker if the saving rate goes from 20% to 40%.

4) Suppose that in US depreciation is 10% and the saving rate is 10%. What is the steady state output per worker in the US ?

5) Now suppose that Chad has the same depreciation rate as US but output per worker in Chad is only 1/20 of US output per worker. What is the saving rate in Chad?

6) (Harder) Compute the golden rule saving rate when the depreciation rate is 10%

7) (Harder) Compute the capital stock per worker that equalize returns to capital and rate of time preference when the depreciation rate is 10% and the rate of time preference R is 110%.

8) (Harder) Compute the saving rate that lead to a steady state capital per worker equal to the one computed in point 7. Compare it to the saving rate computed in 6. Why are they different.

Answers

1) To obtain the per worker productions function divide both sides of the production function by L and obtain

$$\frac{Y}{L} = \frac{K^{.5}L^{.5}}{L} = (\frac{K}{L})^{.5}$$

denoting $y = \frac{Y}{L}$ and $k = \frac{K}{L}$ we can write

$$y = k^{.5} = \sqrt{k}$$

2) To obtain the steady state capital per worker we impose the condition that saving has to be equal to depreciation so

$$s * k_{ss}^{.5} = \delta k_{ss}$$

solving for k and y we obtain

$$
\begin{aligned}
k_{ss} &= (\frac{s}{\delta})^2 \\
y_{ss} &= \sqrt{k_{ss}} = \frac{s}{\delta}
\end{aligned}
$$

since $s = 20\%$ and $\delta = 10\%$ we get $k_{ss} = 4$ and $y_{ss} = 2$.

3) If saving rate goes from 20% to 40% we can use the same formula in point 2 to see that steady state output per worker is equal to 4.

4) If depreciation rate and savings rate are equal to 10% using again the same formulas we find that in US steady state output per worker is equal to 1.

5) Imposing the steady state condition for Chad we obtain

$$y_{ss}^{Chad} = \frac{s^{Chad}}{\delta^{Chad}}$$

solving for the saving rate in Chad we obtain

$$s^{Chad} = y_{ss}^{Chad} * \delta^{Chad}$$

since $y_{ss}^{Chad} = 1/20 * y_{ss}^{US} = 1/20 * 1 = 1/20$ and $\delta^{Chad} = .1$ we obtain that $s^{Chad} = .005 = .5\%$.

6) The golden rule saving rate is the one which maximizes steady state consumption, which is given by

$$c_{ss} = (1 - s)y_{ss} = (1 - s)\frac{s}{\delta}$$

It is immediate to show (using derivatives) that c_{ss} is maximal for $s = 0.5$.

7) The return to capital are given by $mpk + 1 - \delta$ and $mpk = \frac{1}{2}k^{-\frac{1}{2}}$. Equalizing returns to capital and $R=110\%$ one gets

$$k = (2(R - 1 + \delta))^{-2} = \frac{1}{.4^2} = 6.25$$

8) Just use the relation $k_{ss} = (\frac{s}{\delta})^2$ and solve for $s = \delta\sqrt{k_{ss}} = 0.1\sqrt{6.25} = 0.25$. This is smaller than the golden rule saving because consumers have a positive rate of time preference and prefer to anticipate consumption and hence save less than the rate that gives maximal consumption for ever.