University of Minnesota

8107 Macroeconomic Theory, Spring 2012, Mini 1

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Lecture 4. Aggregation with non homothetic preferences and skill heterogeneity

In this class we will consider a simplified version of the model described by Maliar and Maliar (2003) who prove a generalized version of the representative agent result in the case in which there is heterogeneity in skills and in which preferences are not homothetic. The economy is inhabited by a continuum of measure 1 of infinitely lived agents, indexed by i. There is heterogeneity in initial wealth endowments (denoted by k_i) and agents are subject to i.i.d productivity shocks to skills but there is no aggregate uncertainty. Let $\hat{z}_i$ be the skill shock (i.e. shock to its labor endowment) of agent i. Preferences are given by

$$\begin{aligned} & E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^i, 1-l_t^i) \\ u(c, 1-l) &= \frac{c^{1-\sigma}}{1-\sigma} + A\frac{(1-l)^{1-\gamma}}{1-\gamma} \end{aligned}$$

note that these preferences are not homothetic, meaning that wealth expansion paths for c and l are not linear, that is rich and poor agents will choose different mixes of consumption and leisure, unless $\sigma = \gamma$. In general the representative agent result will not hold. We assume complete financial markets, i.e. agents can trade a full set of state-contingent claims (denoted by $b(\hat{z}_{t+1}^i)$) at prices $q(\hat{z}_{t+1}^i)$ that allow them to completely insure against their their own skill shocks. Note that the assumption of no aggregate uncertainty guarantees that these person specific assets are sufficient for all agents to fully insure, i.e. there is no need of assets that are contingent on the aggregate state of the economy. There are also competitive firms who hire effective labor h at rate w and rent capital k at rate r to produce output using a standard CRS technology given by

$$y = k^{\alpha} h^{1-\alpha}$$

The agent's problem can be written as

$$\begin{aligned} & \max_{c_t^i, k_{t+1}^i, l_t^i, b(z_t^i)} E \sum_{t=0}^{\infty} \beta^t u(c_t^i, 1-l_t^i) \\ & s.t. \\ c_t^i + k_{t+1}^i + \int_{z_{t+1}^i} q(\hat{z}_{t+1}^i) b(\hat{z}_{t+1}^i) &= (1-\delta+r_t)k_t^i + w_t \hat{z}_t^i l_t^i + b(\hat{z}_t^i) \\ & b(\hat{z}_0^i), k_0^i \ given \end{aligned}$$

Because of the complete markets assumption and lack of distortions equilibrium allocations in this economy can be characterized using a planning problem which attaches weight λ_i to every agent. It is useful to first define the following variables

$$Z_t = \int_i \hat{z}_t^i, \quad z_t^i = \frac{\hat{z}_t^i}{Z_t}$$

where Z_t is the total skill of the economy at time t and z_t^i is the skill agent i relative to the total; note that $\int z_t^i = 1$. We then define the following aggregate variables:

$$c_t = \int_i c_t^i, \qquad k_t = \int_i k_t^i, \qquad h_t = \int_i z_t^i l_t^i = \frac{1}{Z_t}\int_i \hat{z}_t^i l_t^i$$

and then write the planning problem in two steps. Step 1 is a static problem and is given by

$$\begin{aligned} U(c_t, 1-h_t, \{z_t^i, \lambda^i\}) &\equiv \max_{c_t^i, l_t^i} \int_i \lambda_i u(c_t^i, 1-l_t^i) \\ & \quad s.t \\ c_t &= \int_i c_t^i, \qquad h_t = \int_i z_t^i l_t^i \end{aligned} \tag{1}$$

while step 2 is dynamic and is given by

$$\begin{aligned} & \max_{c_t, h_t, k_{t+1}} E\sum_{t=0}^{\infty}\beta^t U(c_t, 1-h_t, \{z_t^i, \lambda^i\}) \\ & s.t. \\ c_t + k_{t+1} &= Z_t^{1-\alpha}k_t^{\alpha}h_t^{1-\alpha} + (1-\delta)k_t \\ & k_0 \text{ given} \end{aligned}$$

Notice that here U is not (necessarily) the same as u of the individual agent. U represents the indirect utility of the planner of having available today aggregate consumption c_t and aggregate labor input h_t to distribute across agents. Notice that the representative agent result amounts to finding a form for U that i) is known, ii) does not depend (or depends in a simple fashion) on individual level variables i.e. $\{z_t^i, \lambda^i\}$. In order to find the form for U we write the first order conditions of step 1,

$$\begin{aligned} \lambda_i \left(c_t^i\right)^{-\sigma} &= \theta_{1t} \\ \lambda_i A(1-l_t^i)^{-\gamma} &= \theta_{2t} z_{it} \end{aligned}$$

or

$$c_t^i = \left(\frac{\lambda_i}{\theta_{1t}}\right)^{\frac{1}{\sigma}} \tag{2}$$

$$(1-l_t^i)z_t^i = \left(\frac{\lambda_i A}{\theta_{2t}}\right)^{\frac{1}{\gamma}} \left(z_t^i\right)^{\frac{\gamma-1}{\gamma}} \tag{3}$$

where θ_{1t} and θ_{2t} are the Lagrange multipliers on the two constraints in step 1. Next we integrate (2) and (3) across individuals (remember that $\int z_t^i = 1$) to get

$$c_t = \left(\frac{1}{\theta_{1t}}\right)^{\frac{1}{\sigma}} \int_i \lambda_i^{\frac{1}{\sigma}} \tag{4}$$

$$(1-l_t z_t) = 1-h_t = \left(\frac{A}{\theta_{2t}}\right)^{\frac{1}{\gamma}} \int_i \lambda_i^{\frac{1}{\gamma}} \left(z_t^i\right)^{\frac{\gamma-1}{\gamma}} \tag{5}$$

finally dividing c_t^i by c_t and $(1-l_t^i)z_t^i$ by $1-h_t$ we can get

$$
\begin{aligned}
c_t^i &= \frac{\lambda_i^{\frac{1}{\sigma}}}{\int_i \lambda_i^{\frac{1}{\sigma}}} c_t \\
(1-l_t^i) &= \frac{\lambda_i^{\frac{1}{\gamma}} \left(z_t^i\right)^{-\frac{1}{\gamma}}}{\int_i \lambda_i^{\frac{1}{\gamma}} \left(z_t^i\right)^{\frac{\gamma-1}{\gamma}}} (1-h_t)
\end{aligned} \tag{6}
$$

that states that at an efficient allocation individual consumption is a fixed fraction of aggregate consumption (as we have seen in the example in the previous lecture) and individual leisure depends positively on the weight (i.e. richer agents will enjoy more leisure) but negatively on the skill (it is efficient for more skilled agents to enjoy less leisure). The final step to prove the representative agent result is to substitute expressions for c_t^i and $(1-l_t^i)$ into (1), yielding

$$
\begin{aligned}
U(c_t, 1-h_t, \{z_t^i, \lambda^i\}) &= \int_i \lambda_i u(c_t^i, 1-l_t^i) \\
&= \frac{c_t^{1-\sigma}}{1-\sigma} \frac{\int_i \lambda_i \lambda_i^{\frac{1-\sigma}{\sigma}}}{\left(\int_i \lambda_i^{\frac{1}{\sigma}}\right)^{1-\sigma}} + A\frac{(1-h_t)^{1-\gamma}}{1-\gamma} \frac{\int_i \lambda_i \lambda_i^{\frac{1-\gamma}{\gamma}} \left(z_t^i\right)^{-\frac{1-\gamma}{\gamma}}}{\left(\int_i \lambda_i^{\frac{1}{\gamma}} \left(z_t^i\right)^{\frac{\gamma-1}{\gamma}}\right)^{1-\gamma}} \\
&= \frac{c_t^{1-\sigma}}{1-\sigma} \left(\int_i \lambda_i^{\frac{1}{\sigma}}\right)^{\sigma} + A\frac{(1-h_t)^{1-\gamma}}{1-\gamma} \left(\int_i \lambda_i^{\frac{1}{\gamma}} \left(z_t^i\right)^{\frac{\gamma-1}{\gamma}}\right)^{\gamma}
\end{aligned}
$$

since the term $\left(\int_i \lambda_i^{\frac{1}{\sigma}}\right)^{\sigma}$ is constant we can multiply utility by it to get

$$
U(c_t, 1-h_t, \{z_t^i, \lambda^i\}) = \frac{c_t^{1-\sigma}}{1-\sigma} + AX_t \frac{(1-h_t)^{1-\gamma}}{1-\gamma}
$$

where

$$
X_t = \frac{\left(\int_i \lambda_i^{\frac{1}{\gamma}} \left(z_t^i\right)^{\frac{\gamma-1}{\gamma}}\right)^{\gamma}}{\left(\int_i \lambda_i^{\frac{1}{\sigma}}\right)^{\sigma}}
$$

we can also integrate (6) to get

$$
(1-l_t) = Y_t (1-h_t)
$$

where

$$
Y_t = \frac{\int_i \lambda_i^{\frac{1}{\gamma}} \left(z_t^i\right)^{-\frac{1}{\gamma}}}{\int_i \lambda_i^{\frac{1}{\gamma}} \left(z_t^i\right)^{\frac{\gamma-1}{\gamma}}}
$$

notice that we have found sort of RA result in the sense that we can determine the equilibrium path for aggregate variables c_t, h_t, l_t and k_t using "almost" a single agent problem. Notice though that the preferences and technology of the representative agent problem are different from preferences and technology of individuals agents along two important dimensions

i) The utility of the representative agent has a taste shifter X_t which in principle depends on the distribution of skills in the economy. To see this consider the case $\lambda_i = \lambda$ (no wealth heterogeneity) and get

$$
X_t = \left(\int_i \left(z_t^i\right)^{\frac{\gamma-1}{\gamma}}\right)^{\gamma}
$$

where X_t represents the additional (over the aggregate term $(1-h_t)^{1-\gamma}$) utility from leisure coming from heterogeneity. When there is no heterogeneity ($z_t^i = 1$ for all i) $X_t = 1$ and, obviously, there is no additional weight on leisure. Now consider the case in which z is log-normal with parameters given by

$$\log(z) \to N(-\frac{v_t^2}{2}, v_t^2)$$

so that the assumption $\int_i z = 1 = e^{-\frac{v_t^2}{2}+\frac{v_t^2}{2}}$ is satisfied for every t but dispersion v_t can change over time. In this case it is easy to show that

$$X_t = e^{\frac{1-\gamma}{\gamma}\frac{v_t^2}{2}}$$

which shows that how changes in the variance of skills affect the preference for leisure depends on the parameter γ which is related to the elasticity of labor supply. If $\gamma > 1$ (low elasticity of labor supply) an increase in dispersion results in a lower weight on leisure, if $\gamma = 1$ then the weight on leisure is independent on the skill dispersion while if $\gamma < 1$ (high elasticity of labor supply) an increase in skill dispersion results in higher weight on leisure. To see why this is the case think of the value of leisure when, for example, $\gamma < 1$ and v_t is high; in this case any unit of h_t is obtained with relatively low labor effort, as it is efficient to work hard only the highly productive individuals (because of high elasticity) and hence the aggregate value of leisure is high. A preference shock of this type is often referred to as a "labor wedge" (see Chari, Kehoe and McGrattan, 2007) and many authors have shown that such a shock is important to explain business cycles, especially in recent times.

ii) The second dimension along which this model is different from the standard RA is the presence of the "shock" Y_t which relates effective labor h_t to physical labor l_t. As in the previous case we can solve for Y_t in the case of no wealth heterogeneity and under the assumption of log normality of z to get

$$Y_t = e^{-\frac{v_t^2}{\gamma}}$$

which shows that, non surprisingly, the rate at which physical labor is converted in effective labor is increasing in dispersion and in the elasticity of labor supply.

Notice that ne attractive feature of these new shocks X_t and Y_t is that their dynamics do not depend on control variables so they can be determined easily. Notice that if the utility function is homothetic ($\gamma = \sigma$) and no skill heterogeneity is present (i.e. $z_i^t = 1$) then the standard representative agent result applies i.e. preferences of the representative agent are the same as preferences of each individual agent and initial distribution of wealth (summarized by the λ_i) does not matter for aggregates.