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8107 Macroeconomic Theory, Fall 2009, Mini 2

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Lecture 6. Income fluctuations problems, II

In this lecture we want to understand more the dynamics of individual wealth and consumption in income fluctuations problems. In particular we are interested in understanding if the endogenous state space for assets is compact, i.e. there exists an upper bound for assets which is finite or whether there is a tendency for each individual to accumulate assets. This is an important piece for establishing the existence of a stationary equilibrium in economies populated by many agents who face idiosyncratic shocks. As an example think back of the special case of PIH with *i.i.d.* income. and with $\beta(1+r) = 1$.In that case the wealth of an individual which starts with 0 wealth follows a random walk without drift hence it has zero expected value but is unbounded. It follows that the distribution of wealth in an economy with a continuum of agents is not stationary but, since it has 0 mean, it is consistent with general equilibrium.

1 The general problem

We are now going to back to our general problem

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$$
$$s.t.$$
$$c_t + a_{t+1} = (1+r) a_t + y_t$$
$$a_{t+1} \geq -\bar{a}$$

with $u' > 0, u'' < 0$, $(1+r)$ given and $\{y_t\}_{t=0}^{\infty}$ some general stochastic/deterministic process.

The first order necessary condition for optimality is

$$u'(c_t) = \beta(1+r) E_t [u'(c_{t+1})] + \lambda_t, \tag{1}$$

where $\lambda_t > 0$ is the Lagrange multiplier on the borrowing constraint. Condition (1) implies the Euler equation

$$u'(c_t) \geq \beta(1+r) E_t [u'(c_{t+1})]. \tag{2}$$

The value for the interest rate is always *exogenously given.* In general there are going to be two forces shaping asset accumulation. The first is the effective rate of time preference summarized by the term $\beta(1+r)$. Clearly the higher $\beta(1+r)$ the more agents like future conumption the more they'll want to accumulate wealth. The second force is risk. Agents save and accumulate assets to shield their consumption from bad income realizations, so the higher the risk and the the higher is the prudence of agents, the more agents will want to save.

2 Case T finite

If T is finite, obviously c_t and a_t remain bounded, regardless of risk. This is because the effective rate of time preference is set to 0 for ant $t > T$. In general equilibrium, interest rates may exceed the subjective time discount factor, depending on the age profile of labor income. General equilibrium models with many overlapping generations, each of which faces a income fluctuation problem with finite horizon, have become popular tools to analyze policy reforms, from social security reform to fundamental tax reform. Solving the income fluctuation problem in finite horizon numerically is usually fairly easy as straight backward induction can be used.

3 Case T infinite

If T is infinity then the convergence properties of the consumption sequence will depend on the effective rate of time preference $\beta(1+r)$. We always examine three separate cases: $\beta(1+r)$ above, equal to or below one. Also in this case the nature of risk is going to be important. So we first analyze the case of deterministic income fluctuations and then move to stochastic income fluctuations. The cases we are going to analyze with a brief preview of the results are in table 1 below:

Table 1. Wealth dynamics in income fluctuation problems

	Deterministic Income	Stochastic Income
	Wealth dynamics	
$(1+r)\beta > 1$	Diverging	Diverging
$(1+r)\beta = 1$	Stationary	Diverging
$(1+r)\beta < 1$	Stationary	Ambiguous

3.0.1 Deterministic Income Fluctuations

We will focus throughout on the case in which at any point in time net present discounted value of income from that point on is finite i.e.

$$\sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} < \infty \text{ for every } t \tag{3}$$

a special case which obviously satisfies this assumption is a constant income y. In problem set 2 you had to analyze a case in which this assumption is not satisfied.

- Case $\beta(1+r) > 1$: Let's first define the quantity $M_t = u'(c_t)(\beta(1+r))^t$ Without uncertainty the Euler equation (2) implies $M_t \geq M_{t+1} > 0$, which implies that M_t is bounded. Since $\lim_{t\to\infty}(\beta(1+r))^t = \infty$, it must be that $\lim_{t\to\infty} u'(c_t) = 0$, and under regularity conditions for utitility (i.e. Inada) $\lim_{t\to\infty} c_t = \infty$, i.e. consumption is unbounded. Does unbounded consumption implies unbounded assets? It does but one needs to show it. The first step involves the ability of writing the intertemporal budget constraint at every period as

$$a_t(1+r) + \sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} \geq \sum_{j=0}^{\infty} \frac{c_{t+j}}{(1+r)^j}$$

which can be done because $a_{t+1} \geq -\bar{a}$. The second step involves noticing that $\sum_{j=0}^{\infty} \frac{c_{t+j}}{(1+r)^j}$ is unbounded, but $\sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j}$ is bounded by assumption, hence a_t must be unbounded.

- **Case** $\beta(1+r) = 1$**:** From the Euler Equation, $u'(c_t) \geq u'(c_{t+1})$. Households choose perfectly smooth consumption $c_{t+1} = c_t$ when the debt constraint is not binding, otherwise $c_{t+1} > c_t$, so consumption is a *nondecreasing* sequence. One can prove that the effect of the borrowing constraint lasts until a given time τ and vanishes thereafter. Until $t = \tau$, consumption will grow and then it remains constant thereafter. How is τ determined? Define

$$x_t = r \sum_{j=0} \left(\frac{1}{1+r}\right)^j y_{t+j}$$

as the annuity value of the of discounted present value of future income and select τ so that $\bar{x}_\tau = \sup_t x_t$ (which by 3 is guaranteed to exist). Then, consumption will increase until it reaches $\bar{c} = \bar{x}_\tau$ and it will be constant thereafter. So, consumption and assets converge to a finite value. See LS (16.3.1) for a proof of this result.

- **Case** $\beta(1+r) < 1$**:** It should be immediate to see that in this case the consumption and asset sequences will converge as consumers value more early than late consumption. Consider the simple case where the endowment sequence is constant at y and in which $\bar{a} = 0$. Let's write the above problem in DP form with cash-in-hand as a state variable, i.e. $x \equiv (1+r)a + y$. Then:

$$\begin{aligned} V(x) &= \max_{c,a'} \{u(c) + \beta V(x')\} \\ & \quad s.t. \\ x' &= (1+r)(x-c) + y \\ a' &= x - c \\ a' &\geq 0 \end{aligned}$$

From the envelope condition of the above problem,

$$u_c\left(c(x)\right) = V_x\left(x\right),$$

which differentiated w.r.t. to x gives

$$u_{cc}\left(c(x)\right)\frac{dc(x)}{dx} = V_{xx}\left(x\right) \Longrightarrow \frac{dc}{dx} = \frac{V_{xx}\left(x\right)}{u_{cc}\left(c\right)} > 0, \tag{4}$$

thus consumption is increasing in cash-in-hand under concavity and differentiability of V. Next, we want to show that, as long as the borrowing constraint is not binding, cash in hand decreases over time, i.e. if $a'\left(x\right) > 0$ then $x' < x$. When $a'\left(x\right) > 0$, the Euler Equation holds with equality, and:

$$\begin{aligned} u_c\left(c\right) &= \beta\left(1+r\right)V_x\left(x'\right) \\ V_x\left(x\right) &= \beta\left(1+r\right)V_x\left(x'\right) < V_x\left(x'\right) \Longrightarrow x' < x, \end{aligned}$$

where the second line follows from the envelope condition, from $\beta\left(1+r\right) < 1$ and from the concavity of V. Therefore, if we start from a positive level of assets a_0, cash in hand will fall over time and so will consumption because of (4). Next we show that *in finite time* cash in hand converges to y (i.e. assets converge to 0). Suppose not, i.e. that cash in hand converges to a level $\bar{x} > 0$ so that $a(x_t) > 0$ for all x_t. In this case the Euler equation holds with equality for all future periods starting from any date t hence for any t we can write

$$\begin{aligned} 0 &< u'(c_t) \\ &= \lim_{j\to\infty}(\beta(1+r))^j u'(c(x_{t+j})) \\ &\leq \lim_{j\to\infty}(\beta(1+r))^j u'(c(y)) = 0 \end{aligned}$$

which is a contradiction. Note that the inequality in the first row simply follows from the definition of marginal utility, the equality in the second row is derived iterating on the Euler equation and exploiting the fact that it holds with equality for all dates and the last inequality follows from the fact that $x_{t+j} > y$ for all $t+j$ and that consumption in increasing in cash in hand.

Finally, we want to show that once $x = y$ (all assets have been depleted) then $a'\left(x = y\right) = 0$ and $c\left(x\right) = y$, i.e. in the limit as assets get depleted and reach zero (and cash in hand equals y), the consumption sequence converges to the constant endowment stream. We prove it *by contradiction*. Suppose that $a'\left(y\right) > 0$. Then, the FOC holds with equality and

$$\begin{aligned} u_c\left(c(y)\right) &= \beta\left(1+r\right)V_x\left(x'\right) \\ V_x\left(y\right) &= \beta\left(1+r\right)V_x\left(\left(1+r\right)a' + y\right) < V_x\left(\left(1+r\right)a' + y\right) < V_x\left(y\right), \end{aligned}$$

where the second line uses the envelope condition, the fact that $\beta(1+r) < 1$ and the strict concavity of the value function. The second line contains the contradiction.

We conclude that, in the deterministic case, the desire to save is increasing in patience (β) and the interest rate (r). When $\beta(1+r) > 1$ assets and consumption diverge to infinity. When $\beta(1+r) \leq 1$ assets and consumption remain bounded.

3.0.2 Stochastic Income Fluctuations

We now turn to the stochastic case. Here there is an additional motive for saving, the *precautionary motive*, due either to prudence or to the interaction between (i.e., aversion to consumption fluctuations) and the borrowing constraint. It is then intuitive that the condition under which $\{c_t\}$ converges will be more stringent: we will need $\beta(1+r) < 1$.

A useful supermartingale theorem– Multiply both sides of (2) by $\beta^t (1+r)^t$ and define $M_t \equiv \beta^t (1+r)^t u'(c_t) > 0$. Then equation (2) can be written as

$$M_t \geq E_t M_{t+1}.$$

which asserts that M_t follows a *supermartingale.* By the supermartingale converge theorem (Doob, 1995), this (non-negative) stochastic process converges almost surely to a non-negative random variable $\bar{M}$, i.e.,

$$\lim_{t\to\infty} M_t = \bar{M} < \infty \tag{5}$$

in other words the limit is finite.

Case $\beta(1+r) > 1$**:** According to the convergence theorem above, $[\beta(1+r)]^t u'(c_t)$ has a finite limit. Since $[\beta(1+r)]^t \to \infty$, then marginal utility $u'(c_t)$ can only converge to $u'(\bar{c}) = 0$ or, given that $u' > 0$, $c_t \to \infty$. Since debt and income are bounded, $c_t \to \infty$ can only be achieved with $a_t \to \infty$, hence there is no upper bound in the asset space. This is the same result we found for the certainty case.

Case $\beta(1+r) = 1$**:** In this case the most general result is the one provided by Chamberlain and Wilson (2000) who show, using a bounded utility function, that if the process for income is sufficiently volatile consumption and assets will diverge. To see a formal statement of their result see LS, section 16.6. In the special case of *i.i.d.* income process we can prove the result directly. When the income process is *i.i.d.* we can use the cash in hand DP representation of the consumer problem and then, using the Envelope condition into the Euler Equation we get

$$V_x(x_t) \geq E_t[V_x(x_{t+1})],$$

therefore even the derivative of the value function is a super martingale and converges to a *non-negative* random variable. Suppose this limit is strictly positive, then $x_t \to \bar{x}$ finite. But remember that we showed that consumption is a strictly increasing function of cash in hand x (our proof was for the deterministic case but it is straightforward to extend it to the stochstic case). This implies that if x converges to a finite value then also $c(x)$ converges to a finite value. But now consider the budget constraint

$$x' - (1+r)(x-c) = y'$$

we have just shown that the left hand side of the budget constraint converges while the right hand side does not. Hence, the limit cannot be strictly positive and $V_x(x_t) \to 0$ which implies that x_t converges to infinity (why is it?). So the convergence result of the deterministic case does not hold in the stochastic case. We can give a some intuition for this result if we assume that $u''' > 0$. From the Euler Equation

$$u'(c_t) \geq E_t [u'(c_{t+1})].$$

From convexity of the marginal utility, by Jensen's inequality

$$u'(c_t) \geq E_t [u'(c_{t+1})] > u'(E_t(c_{t+1})).$$

By concavity, we have that $E_t(c_{t+1}) > c_t$, so consumption will always tend to ratchet upward over time. The reason for this is exactly the same reason that generates precautionary saving. To see it consider an agent which consumes 1 with certainty today and tomorrow so that its Euler equation holds exactly. Consider now giving the same agent consumption tomorrow equal to 0.9 with 50% probability and 1.1 with 50% probability. If its marginal utility is convex the increase in marginal utility stemming from the 0.9 state is larger than the fall in marginal utility stemming from the 1.1. state so that its expected future marginal utility will increase calling for a transfer of resources from today to tomorrow, i.e. of growing consumption even if $\beta(1+r) = 1$, in other words risk acts as an increase in the rate of time preference β.

Case $\beta(1+r) < 1$. This case is ambiguous. Under additional restriction on utility and on the income process we can prove that the assets are bounded but one cannot prove it in general. Consider the case of *i.i.d.* shocks. Let x be cash in hand. From the Euler Equation:

$$u_c(c(x)) = \beta(1+r)E[u_c(c(x'))] = \beta(1+r)\frac{E[u_c(c(x'))]}{u_c(c(x'_{\max}(x)))}u_c(c(x'_{\max}(x))), \tag{6}$$

where $x'_{\max}(x) = (1+r)(x - c(x)) + y_{\max}$ is the maximum realization of cash in hand next period, given that todays' cash in hand is x. Suppose that the limit

$$\lim_{x\to\infty} \frac{E[u_c(c(x'))]}{u_c(c(x'_{\max}(x)))} = 1. \tag{7}$$

Then, for x large enough, since $\beta(1+r) < 1$, the Euler Equation (6) yields

$$u_c(c(x)) = \beta(1+r)u_c(c(x'_{\max}(x))) < u_c(c(x'_{\max}(x))).$$

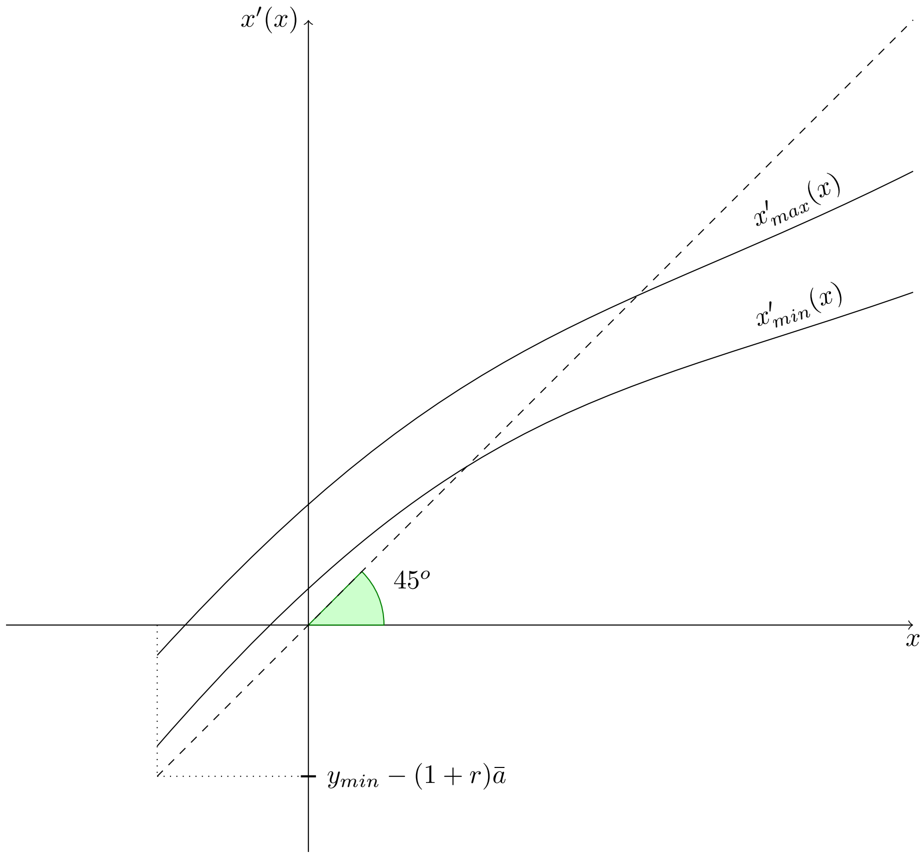

Figure 1: Decision rules for cash in hand, $\beta(1+r) < 1$

Concavity of u implies that

$$c\left(x'_{\max}(x)\right) < c\left(x\right) \Rightarrow x'_{\max}\left(x\right) < x$$

thanks to the fact that $c'\left(x\right) > 0$. And we would be done, because we have demonstrated that cash in hand is bounded. Graphically we'd have established that decision rules look like the one depicted in figure 1.

Therefore, we only need to establish under which condition the limit in (7) holds. To establish this first notice that

$$\frac{u_c(c(x'_{\max}(x) - (y_{\max} - y_{\min})))}{u_c\left(c\left(x'_{\max}(x)\right)\right)} \geqslant \frac{E\left[u_c\left(c\left(x'\right)\right)\right]}{u_c\left(c\left(x'_{\max}(x)\right)\right)} \geqslant 1 \tag{8}$$

where the second inequality simply follows from the fact that marginal utility and the consumption function are increasing and the first inequality can be proven as follows

$$\begin{aligned} E\left[u_c\left(c\left(x'\right)\right)\right] &\leq u_c(c(x'_{\min}(x)) = u_c(c(x'_{\min}(x) + c(x'_{\max}(x)) - c(x'_{\max}(x))) \\ &\leq u_c(c(x'_{\max}(x)) - [c(x'_{\max}(x) - c(x'_{\min}(x))]) \\ &\leq u_c(c(x'_{\max}(x)) - [y_{\max} - y_{\min}]) \end{aligned}$$

where the last line follows from

$$\begin{aligned} c(x'_{\max}(x)) - c(x'_{\min}(x)) &= x'_{\max}(x) - a(x'_{\max}(x)) - (x'_{\min}(x) - a(x'_{\min}(x))) \\ &= y_{\max} - y_{\min} + (a(x'_{\min}(x) - a(x'_{\max}(x))) \\ &\leq y_{\max} - y_{\min} \end{aligned}$$

since $a(x)$ is increasing in x (If a was not increasing in x then we can easily prove that x is bounded). Given (8) for the limit (7) to be true we need

$$\lim_{x\to\infty} \frac{u_c(c(x'_{\max}(x) - (y_{\max} - y_{\min})))}{u_c\left(c\left(x'_{\max}(x)\right)\right)} = 1$$

To show when is this true first note that as $x \longrightarrow \infty, c(x'_{\max}(x)) \longrightarrow \infty$ (this is because concavity of utility implies that as cash in hand goes to ∞ a positive fraction of it goes to current consumption, which thus goes to ∞) then use the following result. If

$$\lim_{c\to\infty} \frac{u_{cc}(c)}{u_c(c)} = 0 \tag{9}$$

i.e. if absolute risk aversion goes to 0 as consumption goes to infinity, then

$$\lim_{c\to\infty} \frac{u_c(c-K)}{u_c(c)} = 1, K > 0$$

To prove the result simply notice that

$$\begin{aligned} \lim_{c\to\infty} \frac{u_c(c-K)}{u_c(c)} &= 1 + \lim_{c\to\infty} \int_0^K \frac{u_{cc}(c-s)}{u_c(c)} ds \\ &= 1 + \lim_{c\to\infty} \int_0^K \frac{u_{cc}(c-s)u_c(c-s)}{u_c(c-s)u_c(c)} ds \\ &\leq 1 + \lim_{c\to\infty} \int_0^K \frac{u_{cc}(c-s)}{u_c(c-s)} ds \\ &= 1 \end{aligned}$$

Notice that this result can be interpreted as marginal utility being close to constant in the limit or in other words that as individuals get rich saving an extra unit has no additional marginal benefit.

To conclude in order to show that assets are bounded one needs *absolute risk aversion that goes to* 0. The intuition is clear: vanishing absolute risk aversion means that the consumer is less worried about income fluctuations as she gets rich, so she will consume more and accumulate less. Note that CRRA utility satisfies condition (9), whereas, obviously, CARA utility does not.

Remember that for this result we require $\beta(1+r) < 1$ and remember that condition on risk aversion is a sufficient condition, i.e. if $\beta(1+r)$ is sufficiently low it might be the case that even with constant absolute risk aversion the asset space is bounded. Finally, recall that this result holds for *i.i.d.* shocks.

Huggett (1993) generalizes this result to a 2-state Markov chain for the income process (with CRRA utility). We conclude by summarizing our findings in:

Result : In presence of borrowing constraints and uncertain income, the condition $\beta(1+r) < 1$ *is necessary for the optimal consumption sequence and for the asset space to be bounded. Moreover, when* $\beta(1+r) < 1$*, if income shocks are iid and absolute risk-aversion goes to 0 as consumption and assets go to infinity then the asset space is bounded.*