University of Minnesota

8107 Macroeconomic Theory, Fall 2009, Mini 2

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Lecture 5. Income fluctuations problems

Our goal is to analyze more general economies in which the representative agent construct does not apply. In particular we will consider an economy in which agents face idiosyncratic income fluctuations against which they can insure using only a non contingent bond. We will first focus on the solution of the individual's problem, given prices and then we will discuss the general equilibrium aspects. These types of economies are often called Bewley/Ayiagari economies.

0.1 General Preferences and Borrowing Constraints

Our problem in its most general form is

$$\begin{aligned} & \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & s.t. \\ c_t + a_{t+1} &= (1+r) a_t + y_t \\ a_{t+1} &\geq -\bar{a} \end{aligned}$$

with $u' > 0, u'' < 0$, $(1+r)$ given and $\{y_t\}_{t=0}^{\infty}$ some general stochastic/deterministic process. How much progress we can make in characterizing the general solution of this problem? To get warmed up with it let's start with some special assumptions.

0.1.1 Quadratic Utility, no borrowing constraints , $\beta(1+r) = 1$

This particular specification of the model is also known as the permanent income hyopthesis (PIH), introduced in economics by Modigliani and Friedman.

We abstract from borrowing constraints and we only impose a No-Ponzi scheme condition stating that in the limit assets cannot be negative, i.e.

$$E_0 \left[\lim_{t \to \infty} \left(\frac{1}{1+r} \right)^t a_t \right] \geq 0,$$

Quadratic utility can be written as

$$u(c) = b_1 c_t - \frac{1}{2} b_2 {c_t}^2,$$

and we also assume that the interest rate on the one-period bond equals the inverse of the discount rate, or $\beta(1+r) = 1$. From the consumption Euler equation:

$$b_1 - b_2 c_t = \beta(1+r) E_t (b_1 - b_2 c_{t+1}) \Rightarrow E_t c_{t+1} = c_t. \tag{1}$$

from which we recover the well known result due to Hall that consumption is a random-walk.Notice that (1) implies

$$E_t c_{t+j} = c_t, \text{ for any } j \geq 0,$$

which is sometimes referred to as the martingale property.

Iterating forward on the budget constraint and using the martingale property and the no-Ponzi condition imply $E_0 \left[\lim_{t \to \infty} \left(\frac{1}{1+r} \right)^t a_t \right] = 0$, we obtain

$$\begin{aligned} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j E_t c_{t+j} &= (1+r)a_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j E_t y_{t+j} \\ c_t &= r a_t + \frac{r}{1+r} \left[\sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j E_t y_{t+j} \right] \end{aligned} \tag{2}$$

Certainty equivalence– Notice that if one consider a different stochastic proces for $\{y_t^*\}_{t=0}^{\infty}$, such that $E_t y_{t+j} = E_t y_{t+j}^*$, for every t, j , then the optimal consumption does not change. Put differently, *the variance and higher moments of the income process do not matter for the determination of consumption.* This property descends from the linear-quadratic objective function. Notice that this property will also in all linearized models.

Consumption change– Now let's define permanent (per period) income $P_t = r a_t + \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j E_t y_{t+j}$ to define the innovation (i.e. the unexpected change) in permanent income, at time t as

$$\begin{aligned} P_{t+1} - E_t P_{t+1} &= r a_{t+1} - E_t \left(r a_{t+1} \right) + \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j \left(E_{t+1} - E_t \right) y_{t+1+j}, \\ &= \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j \left(E_{t+1} - E_t \right) y_{t+1+j}, \end{aligned} \tag{3}$$

where we have used the law of iterated expectations $E_t \left(E_{t+1} x_{t+1+j} \right) = E_t x_{t+1+j}$, and the fact that $r a_{t+1} = E_t \left(r a_{t+1} \right)$, since there is no uncertainty at time t about the evolution of wealth next period

From (2) , the innovation to consumption at time $t+1$ equals

$$\begin{aligned} c_{t+1} - c_t &= c_{t+1} - E_t c_{t+1} = \left[P_{t+1} - E_t P_{t+1} \right], \\ \Delta c_{t+1} &= \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j \left(E_{t+1} - E_t \right) y_{t+1+j}. \end{aligned} \tag{4}$$

where we have used the random walk property and equation (3) . This equation states a useful result:

Result: under the PIH, consumption changes between time t and $t+1$ is proportional to the revision in expected earnings due to the new information accruing in that same time interval.

0.1.2 Example with a Specific Income Process

At this point, to make further progress, we need to make some assumptions on the statistical properties of the labor income process. Let's assume that labor income is a simple autoregressive process

$$y_t = \rho y_{t-1} + \varepsilon_t$$

and let's compute

$$\Delta c_{t+1} = \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j (E_{t+1} - E_t) \, y_{t+1+j}.$$

notice that

$$\begin{aligned} (E_{t+1} - E_t) \, y_{t+1+j} &= E_{t+1} y_{t+1+j} - E_t y_{t+1+j} \\ &= \rho^j y_{t+1} - \rho^{j+1} y_t \\ &= \rho^j (\rho y_t + \varepsilon_{t+1}) - .\rho^{j+1} y_t \\ &= \varepsilon_{t+1} \rho^j \end{aligned}$$

thus

$$\begin{aligned} \Delta c_{t+1} &= \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{\rho}{1+r} \right)^j \varepsilon_{t+1}. \\ &= \frac{r}{1+r-\rho} \varepsilon_{t+1} \end{aligned}$$

this expression is very intuitive and it tells you how much of the innovation to your income process you are going to consume. If $\rho = 1$, $\Delta c_{t+1} = \varepsilon_{t+1}$ that is you are going to consume it all, as you expect it to be fully permanent. On the other hand if $\rho = 0$ you expect it to be temporary so you only consume its annuity value, i.e. $\Delta c_{t+1} = \frac{r}{1+r} \varepsilon_{t+1}$

As another example assume that the income process is the sum of two orthogonal components, a permanent component y_t^p which follows a martingale, and a transitory component u_t that is *iid* over time:

$$\begin{aligned} y_t &= y_t^p + u_t, \\ y_t^p &= y_{t-1}^p + v_t. \end{aligned}$$

(a process which is similar to this is often used in the empirical consumption literature). Using the previous result it is immediate to show that

$$\Delta c_t = \frac{r}{1+r} u_t + v_t.$$

Hence, households adjust their consumption responding to the annuitized change in income. This means that thew will respond only weakly to purely transitory shocks (u_t), whereas they will respond one for one to permanent shocks (v_t). Indeed, the former shocks have only a small effect on permanent income, while the latter change permanent income one for one, by definition.

Identification of the shocks through panel data– Suppose that one has panel data on consumption and income for a sample of households, $i = 1, ..., N$. Let var_i denote the cross-sectional variance. Then, note that

$$\begin{aligned} var_i(\Delta c_t) &= \left(\frac{r}{1+r}\right)^2 var(u_t) + var(v_t) \simeq var(v_t), \\ var_i(\Delta y_t) &= var(v_t) + 2var(u_t), \end{aligned}$$

where the approximate equality holds for r small. Therefore, it is easy to see that with data on consumption and income one can separately identify the variances of the underlying structural income shocks. For example, if over a certain period of time we observe the variance of income rising, but the variance of consumption approximately flat, the PIH model tells us that the rise in income uncertainty was mostly transitory.

0.2 Prudence and Precautionary Savings

In this section we depart from quadratic utility and work with preferences where the marginal utility is nonlinear in order to establish how consumption and saving react to income uncertainty. Unfortunately departing from quadratic utilility and no borrowing constraint reduces a lot our ability of obtaining analytical characterization of the consumption function (One exception that we will discuss later is the case of exponential utility).

0.2.1 Two-period Model

Consider a simple two-period version of the income fluctuations problem discussed above

$$\begin{aligned} &\max_{\{c_0, c_1, a_1\}} u(c_0) + \beta E[u(c_1)] \\ &s.t. \\ c_0 + a_1 &= y_0 \\ c_1 &= (1+r)a_1 + \tilde{y}_1 \end{aligned}$$

where (y_0) is given, and income next period $\tilde{y}_1$ is also exogenous but stochastic. If we retain the assumption $\beta(1+r) = 1$ to simplify the algebra, the Euler equation gives

$$u'(y_0 - a_1) = E[u'((1+r)a_1 + \tilde{y}_1)],$$

which is one equation in one unknown, a_1. The LHS is increasing in a_1 since $u'' < 0$, and the RHS is decreasing because the sum of decreasing functions is a decreasing function, hence a_1^* is uniquely determined.

Mean-preserving spread– What happens to optimal consumption at $t = 0$ if the uncertainty over income next period $\tilde{y}_1$ rises, i.e. as future income becomes more risky? Consider a mean-preserving spread of $\tilde{y}_1$. Define ε to be a random variable with zero mean and positive variance and consider $\hat{y}_1 = \tilde{y}_1 + \varepsilon$. Consier now the right hand side of the Euler equation

$$Eu'((1+r)a_1 + \tilde{y}_1 + \varepsilon.) = E(Eu'((1+r)a_1 + \tilde{y}_1 + \varepsilon.)|\tilde{y}_1) \geq E(u'[(E(1+r)a_1 + \tilde{y}_1 + \varepsilon.)|\tilde{y}_1]) = E(u'[E(1+r)a_1 + \tilde{y}_1])$$

where the first equality uses the law of iterated expectations, the weak inequality follows from the fact that u' *is convex* and from Jensen's inequality and the last equality simply from the definition of conditional mean and from the fact that ε has 0 mean. This shows that a mean-preserving spread of $\tilde{y}_1$ will increase the value of the RHS, i.e it will increase the marginal value of resources tomorrow, which shifts upward, inducing a rise in a_1^* and a fall in c_0^*.

Prudence– The convexity of the marginal utility (or $u''' > 0$) is called "prudence" and is a property of preferences, like risk aversion: risk-aversion refers to the curvature of the utility function, whereas prudence refers to the curvature of the marginal utility function.[1] It can be easily seen that any utility function with decreasing absolute risk aversion (DARA class) displays positive third derivative (e.g., CRRA utility). Intuitively, a rise in uncertainty reduces the certainty-equivalent income next period and with DARA effectively increases the degree of risk-aversion of the agent, inducing him to save more.

Prudence is a motive for additional savings in order to take precaution against possible negative realizations of the income shock next period. In this sense, savings induced by prudence are called "precautionary savings" or "self-insurance". In this simple, two-period partial equilibrium model one can define precautionary wealth due to income uncertainty σ_ε as the difference between the optimal asset choice under uncertainty a_1^* and the optimal asset choice under that would arise when future income is equal to its expcted value with probability 1 . Hence to conclude, we have:

Result: If the marginal utility is convex ($u''' > 0$)*, then the individual is "prudent" and a rise in future income uncertainty leads to a rise in current savings and a decline in current consumption.*

0.2.2 Multi-period case

Let's generalize the two-period model to a multiperiod model with *iid* income shocks and finite-horizon. In the multi-period case (time horizon T), the problem of the household can be written, in recursive form,

[1] Precisely, Kimball (1990) defines the index of relative prudence as the ratio $-[u'''(c)c]/u''(c)$, so in a similar vein to the Arrow-Pratt index of relative risk-aversion $-[u''(c)c]/u'(c)$.

as

$$
\begin{aligned}
V^t(a_t, y_t) &= \max_{\{c_t, a_{t+1}\}} u(c_t) + \beta E\left[V^{t+1}(a_{t+1}, y_{t+1})\right] \\
& \quad s.t. \\
c_t + a_{t+1} &= (1+r)a_t + y_t
\end{aligned}
$$

Note that when the income shocks $\{y_t\}$ are *iid*, we can define a unique state variable which is a sufficient statistics for the household choice, "cash in hand" $x_t = (1+r)a_t + y_t$ since (a_t, y_t) always enter additively and current levels of y_t do not provide any information about the future realizations of income shocks. This leads to the simpler formulation

$$
\begin{aligned}
V^t(x_t) &= \max_{\{c_t, x_{t+1}\}} u(c_t) + \beta E\left[V^{t+1}(x_{t+1})\right] \\
& \quad s.t. \\
x_t &= c_t + a_{t+1} \\
x_{t+1} &= (1+r)(x_t - c_t) + y_{t+1}
\end{aligned}
$$

where the last constraint follows from the definition of cash in hand and the first constraint above:

$$x_{t+1} = (1+r)a_{t+1} + y_{t+1} = (1+r)(x_t - c_t) + y_{t+1}.$$

From the FOC's and the constraints, we obtain (with $\beta(1+r) = 1$)

$$u_1(c_t^*) = E\left[V_1^{t+1}((1+r)(x_t - c_t^*) + y_{t+1})\right], \tag{5}$$

so the precautionary saving result of the two-period model goes through as long as the derivative of the value function V_1^{t+1} is convex, i.e. $V_{111} > 0$. When the time-horizon T is finite, it can be proved that *if* $u_{111} > 0$, *then* $V_{111}^t > 0$ *for all* $t = 0, 1, ..., T$. See Sibley (1975).

0.3 Quadratic Preferences with Borrowing Constraints

Let's go back to the quadratic preference case. So far, we have ignored the presence of borrowing constraints. We imposed a no-Ponzi scheme condition, but we assumed implicitly it's never binding. How restrictive is this abstraction?

Wealth dynamics with borrowing constraints– First, note that, from the budget constraint

$$a_{t+1} - a_t = \Delta a_t = y_t + ra_t - c_t,$$

and using the optimal consumption choice in (2), we obtain

$$\begin{aligned}
\Delta a_t &= y_t + ra_t - ra_t - \frac{r}{1+r}\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j}\right] \\
&= y_t - \frac{r}{1+r} y_t - \frac{r}{1+r}\left[\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j}\right] \\
&= \frac{1}{1+r} y_t - \frac{1}{1+r}\left[(1+r)\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j} - \sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j E_t y_{t+j}\right] \\
&= \frac{1}{1+r} y_t - \frac{1}{1+r}\sum_{j=1}^{\infty}\left[\left(\frac{1}{1+r}\right)^{j-1} E_t y_{t+j} - \left(\frac{1}{1+r}\right)^j E_t y_{t+j}\right] \\
&= \frac{1}{1+r} y_t - \frac{1}{1+r}\left[E_t y_{t+1} + \left(\frac{1}{1+r}\right) E_t y_{t+2} + \left(\frac{1}{1+r}\right)^2 E_t y_{t+3} + ... - \left(\frac{1}{1+r}\right) E_t y_{t+1} - \left(\frac{1}{1+r}\right)^2 E_t y_{t+2} - ...\right] \\
&= -\sum_{j=1}^{\infty}\left(\frac{1}{1+r}\right)^j E_t \Delta y_{t+j}.
\end{aligned}$$

intuitively this equation shows that the growth of assets is negatively related to the future expected growth of income. If an agent expects its income to grow in the future she'll tend to decumulate assets while if she expects income to fall in the future she'll want to accumulate assets. Suppose that the income process follows a random walk, then it is easy to see that $\Delta y_{t+j} = \varepsilon_{t+j}$ and $\Delta a_t = 0$. Therefore, the initial wealth endowment perpetuates itself so if the individual starts above the borrowing constraint, it will never be binding. However, if the income process is $i.i,d.$, we have that $\Delta y_{t+1} = \varepsilon_{t+1} - \varepsilon_t$, $\Delta y_{t+2} = \varepsilon_{t+2} - \varepsilon_{t+1}$,... therefore

$$\Delta a_t = \varepsilon_t,$$

which means that wealth follows a random walk and, as a result, any constraint on asset holdings will be binding with probability one. So, whether the assumption is restrictive or not depends on the income process, but in general this result highlights the fact that borrowing constraints cannot be ignored.

Self-insurance with borrowing constraints– With quadratic preferences, the Euler equation needs to be modified to take into account the possibility that the borrowing constraint is binding. Suppose households face a *no-borrowing constraint* $a_{t+1} \geq 0$. Then, (1) becomes

$$c_t = \begin{cases} E_t c_{t+1} & \text{if } a_{t+1} > 0 \\ y_t + (1+r)a_t & \text{if } a_{t+1} = 0 \end{cases}$$

where the first line is just the FOC of the agent when the constraint is not binding, while the second line descends directly from the budget constraint $a_{t+1} = y_t + a_t - c_t$ when the constraint is binding ($a_{t+1} = 0$). So, if the constraint is binding $c_t = y_t + a_t$, whereas if it is not binding, the agent will save some income and $c_t < y_t + a_t$.

The above condition can be written in compound form as

$$c_t = \min\{y_t + (1+r)a_t, E_t c_{t+1}\} = \min\{y_t + (1+r)a_t, E_t [\min\{y_{t+1} + (1+r)a_{t+1}, E_{t+1}c_{t+2}\}]\} = ...$$

Now, suppose that the uncertainty about income y_{t+1} increases. Low realizations of income y_{t+1} become more likely, which reduces the value of $E_t [\min\{y_{t+1} + (1+r)a_{t+1}, E_{t+1}c_{t+2}\}]$ and makes more likely that the relevant argument of the first min operator is the second argument, thus c_t falls.

Intuitively, when agents face borrowing constraints, they fear getting several consecutive bad income realizations which would push them towards the constraint and force them to consume their income without the ability of smoothing consumption. To prevent this situation, they save for self-insurance (precautionary motive). Thus, we have an important result: prudence is not strictly necessary for precautionary saving behavior, or:

Result: Even in absence of prudence (e.g. with quadratic preferences), in presence of borrowing constraints a rise in future income uncertainty leads to a rise in current savings and a decline in current consumption.