University of Minnesota
8107 Macroeconomic Theory, Fall 2009, Mini 2
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1 Transitions and aggregate uncertainty in economies with idiosyncratic risk and incomplete markets

So far, we have focused on stationary equilibria. However, most policy questions that are analyzed with this class of models involve a full transition path across steady states along which prices change and , as a consequence, individual actions and distributions also change. For example, suppose that we want to study the welfare effect of an *unexpected* permanent rise in the labor income tax rate from τ^* to τ^{**} or to analyze the consequences of a financial liberalization, i.e. a sudden relaxing of the borrowing constraints. Another interesting application is to introduce aggregate fluctuations (i.e. business cycles) in our incomplete markets economy and ask how different the results are from the business cycles results one obtain in a representative agent framework. For example in the representative agent model one famous conclusion (due to Robert Lucas) is that eliminating business cycles has a small welfare effect. Is this conclusion still true in an incomplete markets economy where the representative agent construct no longer applies?

2 Changes in tax policies

To fix ideas in this section we will consider the effects of a change in labor tax policy from τ^* to τ^{**} and suppose that taxes are used to finance government transfers ϕ which are lump-sum. A first way of anlyzing the effects is doing steady state comparisons, i.e. to compute a stationary RCE for these two levels of the tax rate and compare aggregate variables and welfare between the two steady-states. However is unsatifactory at best and misleading in most cases. It can only be used to assess whether a household would prefer to live in the stationary equilibrium of an economy with tax rate equal to τ^* or in the stationary equilibrium of an economy with tax rate equal to τ^{**}, which in most cases is not the relevant comparison. This is an important lesson, one of the few that you should always remember throughout your career as an economist.

Transition– A more interesting and relevant policy question is: consider a household living in the stationary equilibrium of an economy with initial tax rate τ^*. What is the welfare change (gain or loss), for this household, associated to a rise in the tax rate from τ^* to τ^{**}? To answer this question properly, one needs to compute the whole transition: the new policy will change the individual consumption/saving and labor supply decision, hence aggregate prices and will induce dynamics away from the current steady-state towards the new one (assuming the system has stable dynamics). Since the transition is characterized by a sequence of aggregate prices and quantities, we need to modify appropriately the definition of recursive competitive equilibrium.

2.1 Definition of Equilibrium with Transition

First, let's define the household problem at time t still in recursive form

$$\begin{aligned} v_t(a,\varepsilon) &= \max_{c_t,a_{t+1}} \left\{ u(c_t(a,\varepsilon)) + \beta \sum_{\varepsilon_{t+1}\in E} v_{t+1}(a_{t+1}(a,\varepsilon),\varepsilon_{t+1})\pi(\varepsilon_{t+1},\varepsilon) \right\} \\ & \quad s.t. \\ c_t(a,\varepsilon) + a_{t+1}(a,\varepsilon) &= (1+r_t)\, a + w_t\,(1-\tau_t)\,\varepsilon + \phi_t \\ a_{t+1}(a,\varepsilon) &\geq -\bar{a} \end{aligned} \tag{1}$$

Note now that value functions and policies are also a function of time since aggregate prices (r_t, w_t) are time-varying. Let's denote the initial stationary distribution with λ^*.

Given an initial distribution λ^*, and a sequence of tax rates $\{\tau_t\}_{t=0}^{\infty}$, a *competitive equilibrium* is a sequence of value functions $\{v_t\}_{t=0}^{\infty}$ and optimal policies for households $\{c_t, a_{t+1}\}_{t=0}^{\infty}$, optimal firm choices $\{L_t, K_t\}_{t=0}^{\infty}$, prices $\{w_t, r_t\}_{t=0}^{\infty}$, government transfers $\{\phi_t\}_{t=0}^{\infty}$ and distributions $\{\lambda_t\}_{t=0}^{\infty}$ such that, for all t:

- given prices $\{r_t, w_t\}_{t=0}^{\infty}$, the policy functions $a_{t+1}\,(a,\varepsilon)$ and $c_t\,(a,\varepsilon)$ solve the household's problem (1) and $v_t\,(a,\varepsilon)$ is the associated value function,
- given prices $\{r_t, w_t\}_{t=0}^{\infty}$, the firm chooses optimally its capital K_t and its labor L_t, i.e. $r_t + \delta = F_K(K_t, L_t)$ and $w_t = F_H(K_t, L_t)$,
- the labor market clears: $L_t = \int_{A\times E} \varepsilon d\lambda_t(a,\varepsilon) = L$,
- the asset market clears: $K_{t+1} = \int_{A\times E} a_{t+1}(a,\varepsilon) d\lambda_t(a,\varepsilon)$,
- the goods market clears:[1] $\int_{A\times E} c_t(a,\varepsilon) d\lambda_t(a,\varepsilon) + K_{t+1} - (1-\delta)\,K_t = F(K_t, L_t)$,
- the government budget constraint is balanced: $\phi_t = \tau_t w_t L$,
- for all $(\mathcal{A}\times\mathcal{E}) \in \mathcal{S}$, the probability measure λ_{t+1} satisfies

  $$\lambda_{t+1}\,(\mathcal{A}\times\mathcal{E}) = \int_{A\times E} Q_t\,((a,\varepsilon)\,, \mathcal{A}\times\mathcal{E})\, d\lambda_t\,(a,\varepsilon)\,,$$

  where Q_t is the transition function defined as

  $$Q_t\,((a,\varepsilon)\,, \mathcal{A}\times\mathcal{E}) = \sum_{\varepsilon_{t+1}\in\mathcal{E}} I\,\{a_{t+1}(a,\varepsilon)\in\mathcal{A}\}\,\pi(\varepsilon_{t+1},\varepsilon). \tag{2}$$

2.2 Numerical Computation of the Transition Path

The economy at time $t=0$ is in steady-state with stationary distribution $\lambda_0 = \lambda^*$ over assets and individual productivities and tax rate τ^*. At time $t=1$ the government announces that from $t=1$

[1] This condition is redundant by Walras law.

onward the tax policy will change to $\tau^{**} > \tau^*$ and that the additional revenues will augment the lump-sum transfer ϕ_t. Hence, the relevant tax sequence needed to compute the equilibrium is

$$\tau_t = \begin{cases} \tau^*, & \text{for } t = 0 \\ \tau^{**}, & \text{for } t \geq 1. \end{cases}$$

Next, we will assume that after T periods, with T arbitrarily large but finite, the economy will settle to the final steady-state. This assumption is helpful because it allows us to guess a finite sequence of aggregate capital stocks and use backward induction for the solution of the household problem.

To compute the equilibrium follows these steps:

1. Fix T

2. Compute the initial steady state objects $\{v^*, c^*, a^*, K^*\}$ corresponding to $\tau = \tau^*$ and the final steady state objects $\{v^{**}, c^{**}, a^{**}, K^{**}\}$ corresponding to $\tau = \tau^{**}$.

3. Guess a sequence of aggregate capital stocks $\left\{\hat{K}_t\right\}_{t=1}^{T}$ of length T such that $\hat{K}_1 = K^*$ (capital at time 1 is predetermined at time $t = 0$ which is a steady-state) and $\hat{K}_T = K^{**}$. Note that $L_t = L$ (i.e. constant) for every t in absence of endogenous labor supply or aggregate uncertainty. Hence, it is easy to determine, for each t,

$$\begin{aligned} \hat{w}_t &= F_H(\hat{K}_t, L), \\ \hat{r}_t &= F_K(\hat{K}_t, L), \\ \hat{\phi}_t &= \tau_t \hat{w}_t L, \end{aligned}$$

   which are all the elements we need in the budget constraint of the household to solve the household problem at time t.[2]

4. Since we know that $v_T(a, \varepsilon) = v^{**}(a, \varepsilon)$, we can solve the household problem *by backward induction* and derive $\{\hat{v}_t(a, \varepsilon)\}_{t=1}^{T-1}$ from (1) and the associated policy functions $\{\hat{a}_{t+1}(a, \varepsilon)\}_{t=1}^{T-1}$.

5. Given the policy functions, we can reconstruct the sequence of transition functions $\left\{\hat{Q}_t\right\}_{t=1}^{T-1}$ and, since we know that $\lambda_0 = \lambda^*$, we can recover the whole sequence of measures $\left\{\hat{\lambda}_t(a, \varepsilon)\right\}_{t=1}^{T-1}$ and calculate

$$\hat{A}_t = \int_{A \times E} \hat{a}_{t+1}(a, \varepsilon) d\hat{\lambda}_t(a, \varepsilon).$$

6. Check market clearing in the asset market in every period t, i.e. check if the guess of equilibrium capital stocks $\left\{\hat{K}_t\right\}_{t=1}^{T}$ is consistent with aggregate wealth $\left\{\hat{A}_t\right\}_{t=1}^{T}$ that households would accumulate when facing the sequence of prices induced by the proposed sequence of aggregate capital. In other words, choose a convergence criterion ε and check whether

$$\max_{1 \leq t \leq T} |\hat{A}_t - \hat{K}_t| < \varepsilon. \tag{3}$$

[2] In this step, one can equally guess a path for the interest rate or for wages. Then, from the FOC's of the firm, one would recover aggregate capital at each t.

Note that if $|\hat{A}_T - K^{**}| < \varepsilon$ is satisfied, we have implicitly also checked that T is large enough.

7. If inequality (3) is not satisfied at every t, we need a new guess of the capital stock, for example

$$\begin{aligned} \hat{K}_t^{new} &= \hat{K}_t^{old} + \chi \left(\hat{A}_t - \hat{K}_t^{old} \right) \\ & (1-\chi)\hat{K}_t^{old} + \chi \hat{A}_t. \end{aligned}$$

where $0 < \chi \leq 1$ is a parameter that captures how fast we update our guess.

2.3 Computing the Welfare Change from the Tax Reform

Once we have computed the full equilibrium we can ask how much agents gain/lose from the tax reform. Notice that in heterogenoous agents economy welfare analysis is, most of the times, no longer straightforward as it is very hard ro find Pareto improving policies, i.e. policies that improve the welfare of all agents, and many times evaluating a policy will involve evaluating the welfare gains/losses of different groups of the population.

In the first steady-state, an agent with individual state (a, ε) has expected lifetime utility associated with the sequence of optimal consumption choices $\{c_t^*\}_{t=0}^{\infty}$ given by

$$v^*(a,\varepsilon) = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t^*) \,|(a_0 = a, \varepsilon_0 = \varepsilon) \right],$$

where the conditional expectation E_0 is taken over histories of the shocks conditional on a time-zero realization of the shock equal to ε (as made clear by the second equality) and conditional to an agent's wealth level equal to a.

Define the expected discounted utility of an agent with initial state (a, ε) going through the transition as

$$\tilde{v}(a,\varepsilon) = E_0 \left[\sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t) \,|(a_0 = a, \varepsilon_0 = \varepsilon) \right],$$

where $\tilde{v}$ differs from v^* because $\{\tilde{c}_t\}_{t=0}^{\infty}$ is the sequence of optimal consumption choices along the transition path.

Conditional welfare change– The first question we can ask is: how much would an agent with state (a, ε) gain/lose, in percentage terms of lifetime consumption if he lived through the transition, compared to the scenario where he lived in the initial steady-state? Put differently, how much would an agent with wealth-productivity pair (a, ε) in the initial steady-state be willing to give up as a percentage of its lifetime consumption to avoid the tax reform? So, welfare changes are expressed in terms of consumption-equivalent variation.

The answer to this question is a function $\omega(a, \varepsilon)$ that solves the equation

$$E_0 \sum_{t=0}^{\infty} \beta^t u([1+\omega(a,\varepsilon)]\, c_t^*) = E_0 \sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t) = \tilde{v}(a,\varepsilon)$$

For the case of power-utility this calculation is really easy to make. When $u(c) = c^{1-\sigma}$, we can exploit the homogeneity of the utility function and the equation above becomes

$$\begin{aligned} [1+\omega(a,\varepsilon)]^{1-\sigma} v^*(a,\varepsilon) &= \tilde{v}(a,\varepsilon), \\ \omega(a,\varepsilon) &= \left[\frac{\tilde{v}(a,\varepsilon)}{v^*(a,\varepsilon)}\right]^{\frac{1}{1-\sigma}} - 1. \end{aligned} \tag{4}$$

This welfare change is called *conditional welfare change*, because it is computed for an individual that is in a particular state (a,ε). Thus, we can compute the welfare change for the rich household, the poor household, the productive household, the unproductive household, etc...Notice that a given policy (like a tax reduction or increase) can affect different agents in the distribution in very different ways. For example a reduction in the labor tax and a corresponding increase in the lump sum tax makes the agents with high productivity (which have a lot of labor income) happy but it does not make happy agents who have high capital income and low labor income. Conditional welfare pictures are often at the base of political economy models which try to explain why a given policy is or is not adopted.

Ex-ante welfare change– The second type of welfare calculation we can make is one *under the veil of ignorance*, i.e. the one of a planner who weights every agent in the stationary distribution equally. The solution to this welfare calculation, for the power utility case is one number only, ω that solves

$$\omega = \left[\frac{\int_{A\times E} \tilde{v}(a,\varepsilon_i)\, d\lambda^*(a,\varepsilon)}{\int_{A\times E} v^*(a,\varepsilon_i)\, d\lambda^*(a,\varepsilon)}\right]^{\frac{1}{1-\sigma}} - 1.$$

So, ω computes the welfare change for "society", where every agent in society is given equal weight: some will lose, some will gain and we average across everyone with equal weights. Notice that even under equal weights, concavity of the utility function implies that the planner gives a higher weight to poorer agents, i.e. a planner would pick a reform that substract a unit of consumption from a rich agent and give the same unit to a poor agent.

3 Aggregate and Idiosyncratic Risk

We now introduce aggregate fluctuations in the standard Ayiagari economy we have analyzed so far, following the work of Krusell and Smith JPE piece. The key difference is the presence of an aggregate productivity shock z_t that shifts the production function, i.e.

$$Y_t = z_t F(K_t, H_t),$$

and assume that the aggregate shock can take only two values, $z_t \in Z = \{z_b, z_g\}$ with $z_b < z_g$. To keep things simple, we also assume only two values for the individual productivity shock, $\varepsilon_t \in E = \{\varepsilon_b, \varepsilon_g\}$ with $\varepsilon_b < \varepsilon_g$. For example, if $\varepsilon_b = 0$, then it's as if the worker was unemployed for a period. Let

$$\pi(z', \varepsilon' | z, \varepsilon) = \Pr(z_{t+1} = z', \varepsilon_{t+1} = \varepsilon' | z_t = z, \varepsilon_t = \varepsilon)$$

be the Markov chain that describes the joint evolution of the exogenous states. Note that we allow the transition probabilities for ε to depend on z. For example, one should expect that

$$\pi\left(z_b, \varepsilon_g | z_b, \varepsilon_b\right) < \pi\left(z_g, \varepsilon_g | z_b, \varepsilon_b\right), \text{ and } \pi\left(z_b, \varepsilon_g | z_g, \varepsilon_g\right) < \pi\left(z_g, \varepsilon_g | z_g, \varepsilon_g\right)$$

i.e. finding a job is easier if the economy is exiting from a recession and remaining employed is harder if the economy is entering a recession. In our (2×2) case, the Markov transition matrix π has 16 entries.

State variables– The two individual states are $(a, \varepsilon) \in S$ and the two aggregate states are $(z, \lambda) \in Z \times \Lambda$ where $\lambda(a, \varepsilon)$ is the measure of households across states. Note that, although it seems reasonable that λ is enough to complete the description of the state (i.e. a recursive equilibrium exists), this is not obvious at all and there are counterexamples of economies for which one needs to keep track of a longer history of distributions. If you are interested on this there is a good, albeit fairly technical, paper by Duffie, Geanakoplos, Mas-Colell and McLennan (Econometrica, 1994), which provides sufficient conditions for the existence of a recursive equilibrium.

Household Problem– The household problem can be written in recursive form as:

$$\begin{aligned} v(a, \varepsilon; z, \lambda) &= \max_{c, a'} \left\{ u(c) + \beta \sum_{\varepsilon' \in E, z' \in Z} v(a', \varepsilon'; z', \lambda') \pi(z', \varepsilon' | z, \varepsilon) \right\} \qquad (5) \\ & \quad s.t. \\ c + a' &= w(z, K)\varepsilon + (1 + r(z, K)) a \\ a' &\geq 0 \\ K &= \int_{A \times E} a d\lambda \\ \lambda' &= G(z, \lambda, z') \end{aligned}$$

where $G(z, \lambda, z')$ is the law of motion of the endogenous aggregate state and depends on z' since the fraction of agents with $\varepsilon' = \varepsilon_b$ and $\varepsilon' = \varepsilon_g$ next period, given that the current aggregate productivity level is z, depend also on z'.

The key complication is that the value function v depends on λ which is a distribution, i.e. a huge object. Why? To solve the individual problem we need to know prices next period. Next period prices depend on next period aggregate capital, and next period aggregate capital depends on how this period individual saving policies are aggregated through the distribution λ. Thus, to forecast prices, agents need to forecast aggregate capital next period, which is going to depend on λ. Since λ is a state variable, we need to know its equilibrium law of motion G which is an object very difficult for a computer to handle.

A **Recursive Competitive Equilibrium** for this economy is a value function v; policy functions for the household a', and c; policies for the firm H and K; pricing functions r and w; and, a law of motion G such that:

- given the pricing functions $r(z,K)$ and $w(z,K)$, the policy functions a' and c solve the household's problem (5) and v is the associated value function,
- given $r(z,K)$ and $w(z,K)$, the firm chooses optimally its capital K and its labor H, i.e.

$$\begin{aligned} r(z,K)+\delta &= zF_K(K,H), \\ w(z,K) &= zF_H(K,H), \end{aligned} \tag{6}$$

- the labor market clears: $H = \int_{A\times E} \varepsilon d\lambda$,
- the asset market clears: $K = \int_{A\times E} a d\lambda$,
- the goods market clears:

$$\int_{A\times E} c(a,\varepsilon;z,\lambda)d\lambda + \int_{A\times E} a'(a,\varepsilon;z,\lambda)d\lambda = zF(K,H) + (1-\delta)K,$$

- The aggregate law of motion G is generated by the exogenous Markov chain π and the policy function a' as follows:

$$\lambda'(\mathcal{A}\times\mathcal{E}) = G(z,\lambda,z') = \int_{A\times E} Q_{z,z'}((a,\varepsilon),\mathcal{A}\times\mathcal{E})\, d\lambda(a,\varepsilon), \tag{7}$$

  where $Q_{z,z'}$ is the transition function between two periods where the aggregate shock goes from z to z' and is defined by

$$Q_{z,z'}((a,\varepsilon),\mathcal{A}\times\mathcal{E}) = \sum_{\varepsilon'\in\mathcal{E}} I\{a'(a,\varepsilon;z,\lambda)\in\mathcal{A}\}\,\pi(z',\varepsilon'|z,\varepsilon), \tag{8}$$

  where I is the indicator function, and $a'(a,\varepsilon;z,\lambda)$ is the optimal saving policy.

3.1 Computation of an Approximate Equilibrium

The state space of the problem of the household is, technically, infinite-dimensional because it contains a distribution. The reason why we need to keep track of the distribution λ is that agents need to compute next period aggregate capital stock K' to forecast future prices, as clear from (6). Aggregate capital is determined by

$$K' = \int_{A\times E} a'(a,\varepsilon,z,\lambda)d\lambda \tag{9}$$

which is a function of the whole distribution λ, not just of current aggregate K like in a stochastic growth model with a representative agent.[3]

Since we cannot work with an infinitely dimensional distribution, we need to approximate the distribution with a finite-dimensional object. Any distribution can be represented by its entire (in general, infinite) set of moments. Let $\bar{m}$ be a M dimensional vector of the first M moments (mean,

[3]Recall that to solve for the stationary equilibrium, since the distribution is time-invariant, we guess only one value for the capital stock (or the interest rate). To compute the equilibrium transitional dynamics, we guess a sequence of capital stocks. With aggregate uncertainty, we need to guess a law of motion for the aggregate capital stock.

variance, skewness, kurtosis,...) of the wealth distribution, i.e. the marginal of λ with respect to a. Our new state is exactly the vector $\bar{m} = \left\{m^1, m^2, ..., m^M\right\}$ with law of motion

$$\bar{m}' = G_M\left(z, \bar{m}, z'\right). \tag{10}$$

This method is based on the idea that households have partial information about λ. They don't know every detail about that measure, but only a set of moments, e.g. its mean, its variance, the Gini coefficient, the share held by the top 5% and so on. Hence, they use these moments to approximate the true distribution and form forecasts.

Krusell and Smith (1998) have suggested the following implementation of this *partial-information method*. Since prices are only functions of aggregate capital (aggregate labor is exogenously given with inelastic labor supply), one needs only an approximation to equation (9). They suggest one of the form:

$$\ln K' = b_z^0 + b_z^1 \ln K + b_z^2 \left[\ln K\right]^2 + ...,$$

where only the first moment $m^1 = K$ would matter. The new partial-information problem of the agent becomes

$$\begin{aligned} v(a, \varepsilon, z, K) &= \max_{c,a'} \left\{ u(c) + \beta \sum_{\varepsilon', z'} v(a', \varepsilon', z', K') \pi(z', \varepsilon' | z, \varepsilon) \right\} \\ & \quad s.t. \\ c + a' &= w\left(z, K\right) \varepsilon + \left(1 + r\left(z, K\right)\right) a \\ a' &\geq 0 \\ \ln K' &= b_z^0 + b_z^1 \ln K. \end{aligned} \tag{11}$$

Note that now the state space is definitely manageable, we collapsed an infiitely dimensional distribution into one number. Recall that in equilibrium the law of motion of the aggregate state has to be consistent with the aggregation of the optimal individual choices.

The algorithm to solve this problem (and the associated equilibrium) is the following:

1. Guess the coefficients of the law of motion $\left\{b_z^0, b_z^1\right\}$

2. Solve the household problem and obtain the decision rules $a'\left(a, \varepsilon, z, K\right)$, $c\left(a, \varepsilon, z, K\right)$

3. Simulate the economy for I individuals and T periods. For example, $I = 5000$ and $T = 2000$. Draw random sequences for the aggregate shocks and for the individual productivity shocks for each $i = 1, ..I$, conditional on the time-path for the aggregate shocks. Use the decision rules to generate sequences of asset holdings $\left\{a_t^i\right\}_{t=1,i=1}^{T,I}$ and in each period compute the average capital stock

$$K_t = \frac{1}{I} \sum_{i=1}^{I} a_t^i.$$

4. Discard the first T^0 periods (e.g. $T^0 = 500$) to avoid dependence from the initial condition. In the remaining sequence, run the regression

$$\ln K_{t+1} = \beta_z^0 + \beta_z^1 \ln K_t \tag{12}$$

and estimate the coefficients $\left(\beta_z^0, \beta_z^1\right)$.

5. If $\left(\beta_z^0, \beta_z^1\right) \neq \left(b_z^0, b_z^1\right)$, then try a new guess and go back to step 1. If the two pairs are equal for each $z \in \{z_g, z_b\}$, then it means that the approximate law of motion used by the agents is consistent with the one generated in equilibrium by aggregating individual choices. Recall that this equilibrium computation is approximate: we still need to verify how good this approximation is to the fully rational-expectation equilibrium.

6. For this purpose, compute a measure the fit of the regression in step 4), for example by using R^2. Next, try augmenting the state space with another moment, for example using $m^2 = E\left(a_i^2\right)$. Repeat steps 1)-5) until convergence. If the R^2 of the new equation (12) has improved significantly, keep adding moments until R^2 is large and does not respond to addition of new explanatory moments. Otherwise, stop: it means that additional moments do not add new useful information in forecasting prices.

3.2 A Quasi aggregation Result in the Krusell-Smith Economy

Krusell and Smith's main finding is exactly that a law of motion based only on the mean

$$\ln K' = \begin{cases} 0.095 + 0.962 \ln K, & \text{for } z = z_g \\ 0.085 + 0.965 \ln K, & \text{for } z = z_b \end{cases}$$

delivers an $R^2 = 0.999998$ which means that the agents with this simple forecasting rule make very small errors, for example the maximal error in forecasting the interest rate 25 years into the future is around 0.1%.

What is the intuition for the fact that the mean is enough? Recall that if policy functions are of the form

$$a'(a, \varepsilon, z, \lambda) = b_z^0 + b_z^1 a + b_z^2 \varepsilon,$$

then

$$K' = \int_{A \times E} a'(a, \varepsilon, z, \lambda) d\lambda = b_z^0 + b_z^1 K + b_z^2 H_z = \tilde{b}_z^0 + b_z^1 K.$$

But saving functions are concave in general, not linear, so why do we get near-aggregation in practice? The saving functions for this type of problems usually display lots of curvature for low levels of ε and low levels of assets, but beyond this region they're almost linear. Moreover, the agents with this high curvature are few and have low wealth, so they matter very little in determining aggregate wealth.

So, although the agents in the Krusell-Smith economy are prudent (they have CRRA utility), face liquidity constraints which bind with positive probability, and lack full insurance against productivity shocks, the economy in the aggregate behaves almost like a complete-markets economy where aggregation holds perfectly. The reason is that most of the wealth in the economy is held by consumers who can smooth consumption very effectively through self-insurance, hence their saving behavior is guided mostly by their intertemporal motive rather than their insurance motive. Since the intertemporal motive is similar across agents (because similar discount factors and same interest rate) they all react in the same fashion to a productivity shock, hence aggregation applies.

3.3 Quasi aggregation with realistic wealth distributions

The model generates an endogenous consumption and wealth distribution, something that all of representative agents macroeconomics is silent about. Note that the income distribution is, by specifying the income process that households face, an input into the model. To the extent that the income process and hence the cross-sectional income distribution is realistic, one would hope that the resulting wealth distributions (remember that this distribution changes over time) is on average consistent with the cross-sectional wealth distribution in the data. Unfortunately, the model does a fairly bad job reproducing the US wealth distribution, in particular it fails to generate the high concentration of wealth at the upper end of the distribution. In the data, the richest 1% of the US population holds 30% of all household wealth, the top 5% hold 51% of all wealth. For the model described above the corresponding numbers are 3% and 11%, correspondingly. In the model people save to buffer their consumption against unemployment shocks, but since these shocks are infrequent and of short duration, they don't save all that much. Also the model misses on the low wealth housholds. In the US data 11% of households have negative wealth, while in the model this fraction is 0. One might think that this a fairly severe problem and that it drives the quasi aggregation result. Since non linearity of decision rules only shows up at low wealth levels one might think that miss all the low wealth households might drive the quasi-aggregation.

There are several ways of dealing with issue. Krusell and Smith assume that the time discount factor of agents β is stochastic and follows a three state Markov chain with $\beta \in B$ and transition probabilities $\gamma(\beta'|\beta)$. The dynamic programming problem of agents then becomes

$$\begin{aligned} & v(a, y, \beta, s, K) \\ = \quad & \max_{c, a' \geq 0} \left\{ U(c) + \beta \sum_{y' \in Y} \sum_{s' \in S} \sum_{\beta' \in B} \pi(y', s'|y, s) \gamma(\beta'|\beta) v(a', y', \beta', s', K') \right\} \end{aligned} \tag{13}$$

$$s.t.$$

$$c + a' = w(s, K)y + (1 + r(s, K))a \tag{14}$$

$$\log(K') = a_s + b_s \log(K) \tag{15}$$

They pick $B = \{0.9858, 0.9894, 0.993\}$. Hence annual discount rates differ between 2.8% for the most patient agents and 5.6% for the most impatient agents. As transition matrix Krusell and Smith pick a transition which implies that 80% of the population has discount factor in the middle and 10% are on either of the two extremes (in the stationary distribution). It also implies that patient and impatient agents remain so for an expected period of 50 years. De facto this parameterization creates three deterministic types of agents. The patient agents are the ones that will accumulate most of the wealth in this economy. With this trick of stochastic discount factors Krusell and Smith are able to approximate the US wealth distribution to a reasonably accurate degree (see their Figure 3): in the modified economy the richest 1% of the population holds 24% of all wealth, the richest 5% hold 55% percent of all wealth. But most importantly note that the quasi-aggregation result extends to the economy with stochastic discount factors. It is true that now we have agents which operate in the concave part of their saving functions but, by definition, these agents hold a tiny fraction of the aggregate wealth and thus, in term of aggregate dynamics, they do not matter much.

One possible punchline to be learned from this paper is that in terms of aggregate dynamics we can forget about heterogeneity. Although this is true in this special context keep in mind that aggregate dynamics is still driven by black-boxy aggregate productivity shocks and it might well be the case that in order to understand those shocks distributions and heterogeneity are crucial.