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8107 Macroeconomic Theory, Fall 2009, Mini 2

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Stationary equilibria in economies with idiosyncratic Risk and Incomplete Markets

We are now at the point in which we can plug in our individual income fluctuation problem into a general equilibrium framework. The payoff from this is double: first we can analyze aggregate variables such as prices, aggregate employment, business cycles in a truly microfounded fashion. For example many researchers have conjectured that the 2008 crisis has come about because an increase in individual risk (due for example to risk in home prices). This type of models could be used to analyze these issues. The second advantage of such a model is that we'll be able to understand distributions and how, for example, they evolve in response to different type of shocks. The questions we could ask are:

1. How much of the observed wealth inequality can one explain through uninsurable earnings variation across agents?
2. What is the fraction of aggregate savings due to the precautionary motive? (These two questions are analyzed in the paper by Ayiagari)
3. What is the impact of individual risk on the interest rate (This question has been analyzed in the paper by Huggett)
4. What are the welfare consequence of an increase in earnings risk?
5. Who gains and who loses from an increase in sales taxes and a reduction in capital taxes? How do they affect prices?

The economies we will use are constructed around three building blocks: 1) the "income-fluctuation problem" which generate an asset supply , 2) A demand for assets which can be generated in various ways (firms, government etc.) 3) the equilibrium of the asset market. Two key references are the paper by M. Huggett (1993) and the paper by R. Ayiagari (1994) most useful in its working paper version posted on the class page. Also see Ljunqvist Sargent chapter 17 and the Rios-Rull chapter on the Cooley volume. In this note we will focus on stationary equilibria, i.e. on equilibria in which prices are constant through time. In the next lecture we will deal with the case of time or state varying prices.

0.1 Stationary Equilibrium

We will first define a stationary equilibrium through the concept of *Recursive Competitive Equilibrium* (RCE). Most of the requirement of this RCE definition will be standard (agents optimize, markets clear). However, in the stationary equilibrium of this economy we require the distribution of agents across states to be invariant. The probability measure will permanently reproduce itself.

0.1.1 Some Mathematical Preliminaries

The individual is characterized by the pair (a, ε) –the individual states. The aggregate state of the economy is the distribution of agents across states, i.e. $\lambda(a, \varepsilon)$. We would like this object to be a *probability measure,* so we need to define an appropriate mathematical structure. Let $a_{\max}$ be the maximum asset holding in the economy, and for now assume that such upper bound exists. Define the compact set $A \equiv [-\bar{a}, a_{\max}]$ of possible asset holdings and the countable set E be the set of all possible individual shocks. Define the Cartesian product as the state space, $S \equiv A \times E$ with Borel σ algebra $\mathcal{B}$ and typical subset $(\mathcal{A} \times \mathcal{E})$. The space $(S, \mathcal{S})$ is a measurable space, and for any set $\mathcal{S} \in \mathcal{B}$, $\lambda(\mathcal{S})$ is the measure of agents in the set $\mathcal{S}$. Finally let Λ denote the set of all probability measures over $(S, \mathcal{B})$.

How can we characterize the way individuals transit across states over time? i.e. how do we obtain next period distribution, given this period distribution? We need a transition function. Define $Q((a, \varepsilon), \mathcal{A} \times \mathcal{E})$ as the probability that an individual with current state (a, ε) transits to the set $\mathcal{A} \times \mathcal{E}$ next period, formally $Q : S \times \mathcal{B} \to [0, 1]$, and

$$Q((a, \varepsilon), \mathcal{A} \times \mathcal{E}) = \sum_{\varepsilon' \in \mathcal{E}} I\{a'(a, \varepsilon) \in \mathcal{A}\} \pi(\varepsilon', \varepsilon) \tag{1}$$

where I is the indicator function, $a'(a, \varepsilon)$ is the optimal saving policy and $\pi(\varepsilon', \varepsilon)$ is the transition probability function i.e. the probability of having shock ε' tomorrow given that the shock today is ε.Then Q is our transition function and the associated T^* operator yields

$$\lambda_{n+1}(\mathcal{A} \times \mathcal{E}) = T^*(\lambda_n) = \int_{A \times E} Q((a, \varepsilon), \mathcal{A} \times \mathcal{E}) \, d\lambda_n(a, \varepsilon). \tag{2}$$

Let us now re-state the problem of the individual in recursive form,

$$\begin{aligned} v(a, \varepsilon; \lambda) &= \max_{c, a'} \left\{ u(c) + \beta \sum_{\varepsilon' \in E} v(a', \varepsilon'; \lambda') \pi(\varepsilon', \varepsilon) \right\} \qquad (3) \\ & \quad s.t. \\ c + a' &= (1 + r(\lambda)) a + \varepsilon \\ a' &\geq -\bar{a} \end{aligned}$$

where, for clarity, we have made explicit the dependence of prices from the distribution of agents (although, strictly speaking, it is redundant in a stationary environment). We are now ready to proceed to the definition of equilibrium.

0.1.2 Definition of Stationary RCE

A **stationary recursive competitive equilibrium** is a value function $v : S \to \mathbb{R}_+$; policy functions for the household $a' : S \to \mathbb{R}$, and $c : S \to \mathbb{R}_+$; a demand for savings $K(r)$, an interest rate r ; and a stationary measure $\lambda^* \in \Lambda$ such that:

- given r , the policy functions a' and c solve the household's problem (3) and v is the associated value function

- given r the demand for saving result from the optimization of the relevant agents (households, govt, rest of the world)
- the asset market clears: $K(r) = \int_{A\times E} a'(a,\varepsilon)d\lambda^*(a,\varepsilon)$,
- for all $(\mathcal{A}\times\mathcal{E}) \in \mathcal{B}$, the invariant probability measure λ^* satisfies

$$\lambda^*(\mathcal{A}\times\mathcal{E}) = \int_{A\times E} Q\left((a,\varepsilon), \mathcal{A}\times\mathcal{E}\right) d\lambda^*(a,\varepsilon),$$

  where Q is the transition function defined in (1).

0.2 Existence and Uniqueness of the Stationary Equilibrium

Characterizing the conditions under which an equilibrium exists and is unique boils down, like in every general equilibrium model, to show that the excess demand function (of the price) in each market is continuous, strictly monotone and intersects "zero". So in this context if we prove that the equilibrium in the asset market exists and is unique, we are done. If we could show that the aggregate saving supply function

$$A(r) = \int_{A\times E} a'(a,\varepsilon;r)d\lambda^*(a,\varepsilon;r)$$

is continuous in r and crosses the aggregate saving demand function, then we would prove existence. If, in addition, we can show that $A(r)$ is strictly increasing (i.e. in the aggregate the substitution effects outweighs the income effect), we would prove uniqueness. We proceed in steps.

Limits– Suppose first $(1+r)\beta = 1$, i.e. $r = \frac{1}{\beta} - 1$, then we know by the martingale converge theorem that the aggregate supply of assets go to infinity. Strictly speaking in this case there is no stationary distribution hence $A(\frac{1}{\beta}-1)$ is not well defined. Ayiagari argues, invoking a continuity argument, that $\lim_{r\to\frac{1}{\beta}-1} A(r) = +\infty$. For $r = -1$ the individual would like to borrow until the limit, as repayment is costless, so $A(-1) \to -\bar{a}$.[1] In Aiyagari's working paper another form of borrowing constraint is considered, i.e. the case in which $a \geq \frac{\varepsilon_{\min}}{r}$, Given this result if we can prove continuity of $A(r)$ and $K(r)$ we have shown that any model in which $K(-1)-\bar{a} < 0$ and $\lim_{r\to\frac{1}{\beta}-1} K(r) > -\infty$ has at least a stationary equilibrium. Notice that the condition above guarantee that $A(-1)+K(-1) < 0$ and that

$$\lim_{r\to\frac{1}{\beta}-1} A(r) + K(r) > 0$$

so that there exists at least an interest rate r for which the excess demand for saving $A(r) + K(r)$ is 0.For example in the special case of the Huggett model $K(r) = 0$ so that if you prove continuity of $A(r)$ you are done. Another special interesting case is the one presented by Aiyagari in which the demand for capital comes from competitive firms that hire capital and labor to solve the static profict maximization

$$\max_{K.L} K^\alpha L^{1-\alpha} + (1-\delta)K - wL - (1+r)K$$

[1] For values of the interest rate $r < 0$, the agent may still want to hold some wealth for precautionary reasons.

where w is the wage and r is the interest rate. In this case the demand for capital is given by the following expression

$$K(r) = \left(\frac{\alpha L^{1-\alpha}}{\delta + r}\right)^{\frac{1}{1-\alpha}}$$

which tends to $+\infty$ as r goes to $-\delta$ and goes to 0 as r goes to ∞.Also in this case it is easy to verify that continuity of $A(r)$ is sufficient to show existence.

Continuity wrt r– Before we discuss continuity we need to discuss existence of $A(r)$ i.e. we have to establish that a stationary distribution λ^* exist. In particular we have to establish that the operator $T_r^* : * \to *$ defined by

$$\left(T_r^*(\lambda)\right)(\mathcal{A} \times \mathcal{E}) = \int Q_r((a, \varepsilon), (\mathcal{A} \times \mathcal{E}))d\lambda \tag{4}$$

has a unique fixed point (that T_r^* maps $*$ into itself follows from SLP, Theorem 8.2). To show this Aiyagari (in the working paper version, and quite loosely described) draws on a theorem in SLP and Huggett on a similar theorem due to Hopenhayn and Prescott (1992). In both theorems the key condition is a monotone mixing condition that requires a positive probability to go from the highest asset level $a_{\max}$ to a intermediate asset level in N periods and an evenly high probability to go from $-\bar{a}$ assets to an intermediate asset level also in N periods. More precisely stated, the theorem by Hopenhayn and Prescott states the following. Define the order "$\geq$" on S as

$$s \geq s' \text{ iff } [(s_1 \geq s_1' \text{ and } s_2 = s_2') \text{ or } (s' = c = (-\bar{a}, \varepsilon_1)) \text{ or } (s = d = (a_{\max}, \varepsilon_N)) \tag{5}$$

Under this order it is easy to show that $(S, \geq)$ is an ordered space, S together with the Euclidean metric is a compact metric space, $\geq$ is a closed order, $c \in S$ and $d \in S$ are the smallest and the largest elements in S (under order $\geq$) and $(S, \mathcal{B}(S))$ is a measurable space. Then we have (see Hopenhayn and Prescott (1992), Theorem 2)

Proposition 1 *If*

1. *Q_r is a transition function*
2. *Q_r is increasing*
3. *There exists $s^* \in S$, $\delta > 0$ and N such that*

$$P^N(d, \{s : s \leq s^*\}) > \delta \text{ and } P^N(c, \{s : s \geq s^*\}) > \delta \tag{6}$$

Then the operator T_r^* has a unique fixed point λ_r and for all $\lambda_0 \in \Lambda$ the sequence of measures defined by

$$\lambda_n = (T^*)^n \lambda_0 \tag{7}$$

converges weakly to λ_r

Here $P^N\{s, \mathcal{S})$ is the probability of going from state s to set $\mathcal{S}$ in N steps. Instead of proving this result (which turns out to be quite tough) we will explain the assumptions, heuristically verify them and discuss what the theorem delivers for us. Assumption 1 requires that Q_r is in fact a transition function, i.e. $Q_r(s,.)$ is a probability measure on $(S,\mathcal{B}(S))$ for all $s \in S$ and $Q_r(.,\mathcal{S})$ is a $\mathcal{B}(S)$-measurable function for all $\mathcal{S} \in \mathcal{B}(S)$. Given that $a'(a,y)$ is a continuous function, the proof of this is not too hard. The assumption that Q_r is increasing means that for any nondecreasing function $f : S \to \mathbf{R}$ we have that

$$(Tf)(s) = \int f(s')Q_r(s,ds') \tag{8}$$

is also nondecreasing. The proof that Q_r satisfies monotonicity is straightforward, given that $a'(a,y)$ is increasing in both its arguments[2], so that bigger s's make $Q_r(s,.)$ put more probability mass on bigger s'. Together with f being nondecreasing the result follows. Finally, why is the monotone mixing condition 3 satisfied? Loosely speaking, suppose the household starts from $(\bar{a},\varepsilon_{\max})$ and receives a long stream of the worse realization of the shock $\varepsilon_{\min}$. Then, she will keep decumulating wealth until she reaches some neighborhood of the lower bound. The reason for decumulation is that the household knows that this income realization is well below average, his permanent income is higher and consumption is dictated by permanent income. Suppose now that the household starts with $(-\bar{a},\varepsilon_{\min})$ and receives a long stream of the best shock $\varepsilon_{\max}$. Then, she will accumulate wealth until she reaches some neighborhood of the upper bound. The reason for accumulation is similar: the household realizes that this good realization is "transitory" and her expected income is below the current income, so she saves a fraction of these lucky draws.

The conclusion of the theorem then assures the existence of a unique invariant measure λ_r which can be found by iterating on the operator T_r^*. Convergence is in the weak sense, that is, a sequence of measures $\{\lambda_n\}$ converges weakly to λ_r if for every continuous and bounded real-valued function f on S we have

$$\lim_{n\to\infty} \int f(s)d\lambda_n = \int f(s)d\lambda_r \tag{9}$$

The argument in the preceding section demonstrated that the function $Ea(r)$ is well-defined on $r \in [-1, \frac{1}{\beta} - 1)$. Since $a'_r(a,y)$ is a continuous function jointly in (r,a), see SLP, Theorem 3.8 and λ_r is continuous in r (in the sense of weak convergence), see SLP, Theorem 12.13, the function $Ea(r)$ is a continuous function of r on $[-1,\rho)$. Note that the real crux of the argument is in establishing an upper bound of the state space for assets, as S needs to be compact for the theorem by Hopenhayn and Prescott to work. To bound the state space an interest rate $r < \rho$ is needed, as we have discussed in previous classes.

Monotonicity– There are no results on the monotonicity of the aggregate supply of capital with respect to r, so uniqueness is never guaranteed. One can use the computer to plot aggregate supply as a function of the interest rate on a fine grid for a reasonably large range of values of r to check its slope. But notice that even if the aggregate supply of capital is monotone there could be multiple equilibria if the demand for capital is increasing or non monotone. Sargent and Ljunqvist in section 17.11 describe an economy with seignorage in which typically there are multiple stationary equilibria.

[2]For $a'(a,\varepsilon)$ to be increasing in y we need to assume that the exogenous Markov chain does not feature negatively serially correlated income.

0.3 An Algorithm for the Computation of the Equilibrium

How do we compute, in practice, this equilibrium? The algorithm that can be used is a fixed point algorithm over the interest rate. Here we give an example of this algorithm for the Ayiagari economy

1. Fix an initial guess for the interest rate $r^0 \in \left(-\delta, \frac{1}{\beta} - 1\right)$, where these bounds follow from our previous discussion. The interest rate r^0 is our first candidate for the equilibrium (the superscript denotes the iteration number).

2. Given the interest rate r^0, obtain the wage rate $w(r^0)$ using the CRS property of the production function (recall that H is given exogenously with inelastic labor supply).

3. Given prices $\left(r^0, w(r^0)\right)$, you can now solve the dynamic programming problem of the agent (3) to obtain $a'(a, \varepsilon; r^0)$ and $c(a, \varepsilon; r^0)$.

4. Given the policy function $a'(a, \varepsilon; r^0)$ and the Markov transition over productivity shocks $\pi(\varepsilon', \varepsilon)$, we can construct the transition function $Q\left(r^0\right)$ and obtain the fixed point distribution λ_{r^0}, conditional on the candidate interest rate r^0.

   (a) The easiest method to implement this step, in practice, is to approximate a distribution λ_{r^0} with a M dimensional column vector and the transition function $Q\left(r^0\right)$ with a $M * M$ matrix $Q^M_{r^0}$ which represents a linear map from the space of M dimensional vectors into itself so that given a distribution λ_{r^0} today the distribution tomorrow is given by $Q^M_{r^0}$ λ_{r^0}. After appropriately constructing the matrix $Q^M_{r^0}$ using the numerically computed decision rules together with the Markov chain approximation for the stochastic process the stationary distribution $\lambda^*_{r^0}$ can be simply computed by finding the, appropiately normalized, eigenvector associated to the unitary eigenvalue of $Q^M_{r^0}$.

5. Compute the aggregate demand of capital $K\left(r^0\right)$ from the optimal choice of the firm who takes as given r^0, i.e.
$$K(r^0) = F_k^{-1}(r^0 + \delta)$$

6. Compute the integral
$$A\left(r^0\right) = \int_{A \times E} a'(a, \varepsilon; r^0) d\lambda^*_{r^0}$$
which gives the aggregate supply of assets. .

7. Compare $K\left(r^0\right)$ with $A\left(r^0\right)$ to verify the asset market clearing condition. If $A\left(r^0\right) > (<) K\left(r^0\right)$, then the next guess of the interest rate should be lower (higher), i.e. $r^1 < (>) r^0$. To obtain the new candidate r^1 a good choice is, for example,
$$r^1 = \frac{1}{2}\left\{r^0 + \left[F_K(A\left(r^0\right), H) - \delta\right]\right\}$$

8. Update your guess to r^1 and go back to step 1). Keep iterating until one reaches convergence of the interest rate, i.e. until
$$|r^{n+1} - r^n| < \varepsilon,$$

for ε small.

9. All the equilibrium statistics of interest, like aggregate savings, inequality measures, etc. can be then easily computed using the stationary distribution.