University of Minnesota

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Lecture 5. Income fluctuations problems II

Consider again the general problem

$$\begin{aligned} & \max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \\ & s.t. \\ c_t + a_{t+1} &= (1+r) a_t + y_t \\ a_{t+1} &\geq -\bar{a} \end{aligned}$$

with $u' > 0, u'' < 0$, $(1+r)$ given and $\{y_t\}_{t=0}^{\infty}$ some general stochastic/deterministic process.The first order necessary condition for optimality is

$$u'(c_t) = \beta(1+r) E_t[u'(c_{t+1})] + \lambda_t, \tag{1}$$

where $\lambda_t > 0$ is the Lagrange multiplier on the borrowing constraint. Condition (1) implies the Euler equation

$$u'(c_t) \geq \beta(1+r) E_t[u'(c_{t+1})]. \tag{2}$$

The value for the interest rate is for now *exogenously given.* We want to understand whether the optimal consumption sequence $\{c_t\}$ is bounded above or whether it will be diverging as $t \to \infty$. This characterization is important for the purposes of estabilishing the existence and for characterizing an equilibrium in economies populated by many agents who face idiosyncratic shocks and each choose her optimal consumption by solving her own income fluctuation problem. If the consumption sequence is bounded, then the endogenous state space for assets $[-\bar{a}, a_{MAX}]$ is compact, i.e. there exists an upper bound a_{MAX} which is finite. This gurantees that the asset demand from all households is finite and thus r could be a potentiall equilibrium interest rate. Alternatively, if the consumption sequence diverges, then $a_{MAX} \to \infty$. This means that, in such an economy, there will be an infinite demand for assets and no equilibrium can be found at that given interest rate.

0.1 Case T finite

If T is finite, obviously c_t and a_t remain bounded. In general equilibrium, interest rates may exceed the subjective time discount factor, depending on the age profile of labor income. General equilibrium models with many overlapping generations, each of which faces a income fluctuation problem with finite horizon, have become popular tools to analyze policy reforms, from social security reform to fundamental tax reform. Solving the income fluctuation problem in finite horizon numerically is usually fairly easy as straight backward induction can be used.

0.2 Case T infinite

If T is infinity then the convergence properties of the consumption sequence will depend on the value of $\beta(1+r)$. We always examine three separate cases: $\beta(1+r)$ above, equal to or below one. We start from a problem where income fluctuations are deterministic, i.e., perfectly foreseen. Next we move to stochastic income fluctuations.

0.2.1 Deterministic Income Fluctuations

For alll these cases we will focus on the case in which at any point in time net present discounted value of income from that point on is finite i.e.

$$\sum_{j=0}^{\infty} \frac{y_{t+j}}{(1+r)^j} < \infty \quad \text{for every } t$$

a special case which obviously satisfies this assumption is a constant income y.

Case $\beta(1+r) > 1$**:** Let's first define the quantity $M_t = u'(c_t)(\beta(1+r))^t$ Without uncertainty the Euler equation (2) implies $M_t \geq M_{t+1} > 0$, which implies that M_t is bounded. Since $\lim_{t\to\infty}(\beta(1+r))^t = \infty$, it must be that $\lim_{t\to\infty} u'(c_t) =0$, and under regularity conditions for utitility (i.e. Inada) $\lim_{t\to\infty} c_t = \infty$, i.e. consumption is unbounded. Does unbounded consumption implies unbounded assets? It does but one needs to show it. The first step involves the ability of writing the intertemporal budget constraint at every t as

$$a_t(1+r) + \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j y_{t+j} \geq \sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j c_{t+j}$$

which can be done because $a_{t+1} \geq -\bar{a}$. The second step involves noticing that $\sum_{j=0}\left(\frac{1}{1+r}\right)^j c_{t+j}$ is unbounded, but $\sum_{j=0}\left(\frac{1}{1+r}\right)^j y_{t+j}$ is bounded by assumption, hence a_t must be unbounded.

Case $\beta(1+r) = 1$**:** From the Euler Equation, $u'(c_t) \geq u'(c_{t+1})$. Households want perfectly smooth consumption $c_{t+1} = c_t$ when the debt constraint is not binding, otherwise $c_{t+1} > c_t$, so consumption is a *nondecreasing* sequence. One can prove that the effect of the borrowing constraint lasts until a given time τ and vanishes thereafter. Until $t = \tau$, consumption will grow and then it remains constant thereafter. How is τ determined? Define

$$x_t = r\sum_{j=0}\left(\frac{1}{1+r}\right)^j y_{t+j}$$

as the annuity value of the of discounted present value of future income and select τ so that $\bar{x}_\tau = \sup_t x_t$. Then, consumption will increase until it reaches $\bar{c} = \bar{x}_\tau$ and it will be constant thereafter. So, using the same logic as before, consumption and assets converge to a finite value. See LS (16.3.1) for a complete proof of this result in the special case in which $\bar{a} = 0$, i.e. no borrowing is allowed.

Case $\beta(1+r) < 1$**:** It should be immediate that in this case the consumption and assets will converge. Consider the simple case where the endowment sequence is constant at y. Let's write the above problem in DP form with cash-in-hand as a state variable, i.e. $x \equiv (1+r)\,a + y$. Then:

$$\begin{aligned} V(x) &= \max_{c,a'} \{u(c) + \beta V(x')\} \\ & \quad s.t. \\ c + a' &= x \\ x' &= (1+r)(x-c) + y \\ a' &\geq 0 \end{aligned}$$

From the envelope condition of the above problem,

$$u_c(c(x)) = V_x(x),$$

which differentiated w.r.t to x gives

$$u_{cc}(c(x))\frac{dc}{dx} = V_{xx}(x) \Longrightarrow \frac{dc}{dx} = \frac{V_{xx}(x)}{u_{cc}(c)} > 0, \tag{3}$$

thus consumption is increasing in cash-in-hand under concavity of V.

Next, we want to show that, as long as the borrowing constraint is not binding, cash in hand decreases over time, i.e. if $a'(x) > 0$ then $x' < x$. When $a'(x) > 0$, the Euler Equation holds with equality, and:

$$\begin{aligned} u_c(c(x)) &= \beta(1+r)V_x(x') \\ V_x(x) &= \beta(1+r)V_x(x') < V_x(x') \Longrightarrow x' < x, \end{aligned}$$

where the second line follows from the envelope condition, from $\beta(1+r) < 1$ and from the concavity of V. Therefore, if we start from a positive level of assets a_0, cash in hand will fall over time and so will consumption because of (3). The next thing one needs to show is that *in finite time* cash in hand converges to y (i.e. assets converge to 0). Suppose not, i.e. that cash in hand converges to a level $\bar{x} > y$ so that $a'(x_t) > 0$ for all x_t. In this case the Euler equation holds with equality for all future periods starting from any date t hence for any t we can write

$$\begin{aligned} 0 &< u'(c_t) \\ &= \lim_{j\to\infty}(\beta(1+r))^j u'(c(x_{t+j})) \\ &\leq \lim_{j\to\infty}(\beta(1+r))^j u'(c(y)) = 0 \end{aligned}$$

which is a contracdiction. Note that the inequality in the first row simply follows from the definition of marginal utility, the equality in the second row follows from the fact that the Euler equation holds with equality for all dates and the last inequality follows from the fact that $x_{t+j} > y$ for all $t+j$ and that consumption in increasing in cash in hand.

Finally, we want to show that once $x = y$, (i.e. when all assets have been depleted) then $a'(x) = 0$ and $c(x) = y$. Again we prove it *by contradiction.* Suppose that $a'(y) > 0$. Then, the FOC holds with equality and

$$\begin{aligned} u_c(c(y)) &= \beta(1+r)V_x(x') \\ V_x(y) &= \beta(1+r)V_x((1+r)a' + y) < V_x((1+r)a' + y) < V_x(y), \end{aligned}$$

where the second line uses the envelope condition, the fact that $\beta(1+r) < 1$ and the strict concavity of the value function. The second line contains the contradiction.

We conclude that, in the deterministic case, the desire to save is increasing in patience (β) and the interest rate (r). When $\beta(1+r) > 1$, both assets and consumption diverge to infinity. When $\beta R \leq 1$ assets and consumption will remain bounded (provided that the income process remains bounded).

0.2.2 Stochastic Income Fluctuations

We now turn to the stochastic case. Our benchmark model for income is going to be a Markov chain, i.e. we are going to assume that y_t can take a finite (N) number of values and that the transition probabilities from one value of y today to abother value of y tomorrow can be described by a matrix. In this enviroment there is an additional motive for saving, the *precautionary motive*, due to the interaction between risk-aversion (i.e., aversion to consumption fluctuations) and the borrowing constraint. It is then intuitive that the condition under which assets and consumption converge will be more stringent: we will need $\beta R < 1$ and even that it will not be a sufficient condition.

A useful supermartingale– Multiply both sides of (2) by $\beta^t (1+r)^t$ and define $M_t \equiv \beta^t (1+r)^t u'(c_t) > 0$. Then equation (2) can be written as

$$M_t \geq E_t M_{t+1}.$$

which asserts that M_t follows a *supermartingale.* By the supermartingale converge theorem (Doob, 1953, see LS p. 560 for a precise statement of the theorem), this (non-negative) stochastic process converges almost surely to a non-negative random variable $\bar{M}$, i.e.,

$$\lim_{t\to\infty} M_t = \bar{M} < \infty \tag{4}$$

in other words the limit is finite.

Case $\beta(1+r) > 1$**:** According to the convergence theorem above, $[\beta(1+r)]^t u'(c_t)$ has a finite limit. Since $[\beta(1+r)]^t \to \infty$, then marginal utility $u'(c_t)$ can only converge to $u'(c_t) = 0$ or, given the Inada condition, $c_t \to \infty$. Since debt is limited and income is bounded, divergence of consumption means $a_t \to \infty$, hence there is no upper bound in the asset space. This is the same result we found for the certainty case. Note that if the utility function does not satisfy the Inada condition (for example quadratic utility) then it is not necessarily the case that assets and consumption diverge.

Case $\beta(1+r) = 1$**:** In this case the most general result is the one provided by Chamberlain and Wilson (2000) who show, using a bounded utility function, that if the process for income is sufficiently volatile consumption and assets will diverge. To see a formal statement of their result see LS, section 16.6. In the special case of i.i.d income process we can prove the result directly.

When the income process is i.id we can use the cash in hand DP representation of the consumer problem and then, using the Envelope condition into the Euler Equation we get

$$V_x(x_t) \geq E_t[V_x(x_{t+1})],$$

showing that the derivative of the value function is a super martingale and converges to a *non-negative* random variable. Suppose this limit is strictly positive, then $x_t \to \bar{x}$ finite. But remember that we showed that consumption is a strictly increasin function of cash in hand x. This implies that if x converges to a finite value then also $c(x)$ converges to a finite value. But now consider the budget constraint

$$x' - (1+r)(x-c) = y$$

we have just shown that the left hand side of the budget constraint converges while the right hand side does not. Hence, the limit cannot be strictly positive and $V_x(x_t) \to 0$ which implies that $x_t \to \infty$. So, the asset space is unbounded. Another proof of the same result is obtained by contradiction. Assume that there is a value of cash in hand $x_{\max}$, such that $\max_{y'} x'(x_{\max}) = \max_{y'}(1+r)a(x_{\max}) + y' \leq x_{\max}$, that is there is a value for cash in hand at which cash in hand is decreasing for all possible future shocks (this implies that cash in hand will never go above that value and thus is bounded). Then use Euler equation and envelope condition, together with the fact that $\beta R = 1$ to get

$$V_x(x_{\max}) \geq E_t[V_x(x'(x_{\max}))]$$

using then the strict concavity of the value function yields

$$E_t[V_x(x'(x_{\max}))] > V_x(\max_{y'} y' + (1+r)a(x_{\max}))$$

and finally using the assumption on cash in hand

$$V_x(\max_{y'} y' + (1+r)a(x_{\max})) \geq V_x(x_{\max})$$

which gives a contradiction. We can get a bit of intuition for this result if we assume that $u''' > 0$. From the Euler Equation

$$u'(c_t) \geq E_t[u'(c_{t+1})].$$

From convexity of the marginal utility, by Jensen's inequality

$$u'(c_t) \geq E_t[u'(c_{t+1})] > u'(E_t(c_{t+1})).$$

By concavity, we have that $E_t(c_{t+1}) > c_t$, so consumption will always tend to ratchet upward over time. The reason for this is exactly the same reason that generated precautionary saving. To see it consider an agent which consumes 1 with certainty today and tomorrow so that its Euler equation holds exactly. Consider now giving the same agent consumption tomorrow equal to 0.9 with 50% probability and 1.1 with 50% probability. If its marginal utility is convex the increase in marginal utility stemming from the 0.9 state is larger than the fall in marginal utility styemming from the 1.1. state so that its expected future marginal utility will increase calling for a transfer of resources from today to tomorrow, i.e. of growing consumption

Case $\beta(1+r)<1$**:** Consider first the case of *iid* shocks. Let x be cash in hand. From the Euler Equation:

$$u_c(c(x)) = \beta RE[u_c(c(x'))] = \beta R \frac{E[u_c(c(x'))]}{u_c(c_{\max}(x))} u_c(c_{\max}(x)), \tag{5}$$

where $c_{\max}(x) = c(x'_{\max}) = c(Ra'(x) + y_{\max})$ is the consumption associated to the maximum realization of cash in hand next period, given that todays' cash in hand is x. Suppose that the limit

$$\lim_{x\to\infty} \frac{E[u_c(c(x'))]}{u_c(c_{\max}(x))} = 1. \tag{6}$$

Then, for x large enough, since $\beta(1+r)<1$, the Euler Equation (5) yields

$$u'(c(x)) = \beta Ru'(c_{\max}(x)) < u'(c_{\max}(x)).$$

Concavity of u implies that

$$c_{\max}(x) < c(x) \Rightarrow Ra'(x) + y_{\max} < x \Rightarrow x'_{\max}(x) < x$$

thanks to the fact that $c'(x)>0$. And we would be done, because we have demonstrated that cash in hand does not increase forever. Therefore, we only need to establish under which condition the limit in (6) holds.

Consider $u_c(c(x'))$ and compute a first-order Taylor approximation around $x' = x'_{\max}$:

$$u_c(c(x')) \simeq u_c(c(x'_{\max})) + u_{cc}(c(x'_{\max}))\, c_x(x'_{\max})(x' - x'_{\max}).$$

Taking expectations of both sides

$$E[u_c(c(x'))] \simeq u_c(c(x'_{\max})) - u_{cc}(c(x'_{\max}))\, E[x'_{\max} - x']\, c_x(x'_{\max})$$

and dividing by $u_c(c_{\max}(x))$:

$$\frac{E[u_c(c(x'))]}{u_c(c_{\max}(x))} \simeq 1 - \frac{u_{cc}(c_{\max}(x))}{u_c(c_{\max}(x))} E[y_{\max} - y']\, c_x(x'_{\max}) = 1 + \sigma_{ABS}(c_{\max}(x))[y_{\max} - E(y')]\, c_x(x'_{\max}),$$

where σ_{ABS} is the coefficient of absolute risk aversion. Assuming $[y_{\max} - E(y')]$ and $c_x(x'_{\max})$ are positive and finite, the key condition that we need to verify is

$$\lim_{x\to\infty} \sigma_{ABS}(c(x)) = 0. \tag{7}$$

In other words, we need *absolute risk aversion to be decreasing with asset holdings*. The faster it decreases, the smaller is the upper bound on the asset space. The intuition is clear: DARA means that the agent is less worried about income fluctuations as she gets rich, so she will consume more and accumulate less. Note that CRRA utility has DARA, so it satisfies condition (7), whereas, obviously, CARA utility fails to satisfy it.

Remember that for this result we require $\beta(1+r)<1$ and remember that DARA is a sufficient condition. Finally, recall that this result holds for *iid* shocks. Huggett (1993) generalizes this result to a 2-state Markov chain for the income process (with CRRA utility). We conclude by summarizing our findings as follows: *In presence of borrowing constraints and uncertain income, the condition* $\beta(1+r)<1$ *is necessary for the optimal consumption sequence and for the asset space to be bounded. Moreover, when* $\beta(1+r)<1$*, if income shocks are iid and absolute risk-aversion is decreasing (DARA utility), then the asset space is bounded.*