University of Minnesota

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Lecture 2. Theory of distributions with representative consumers

In this lecture we consider a dynamic economy in which there is heterogeneity in wealth endowments and in which the representative agent result applies. Our focus is on how aggregate dynamics affect the dynamics of the wealth and consumption distribution (although the wealth distribution and its dynamics do not affect aggregate dynamics). This is probably the simplest framework within which we can study the determination and the evolution of distributions, such as the wealth distribution, the income distribution or the consumption distribution. Key references that you should read are the papers by Chatterjee (JPUBE, 1994) and Caselli and Ventura (AER 2000) which are available on the class page.

1 The economy

Demographics and preferences– The economy is inhabited by N types of infinitely lived agents, indexed by $i = 1, 2, ..., N$. Denote by μ^i the measure of agents i and normalize the total measure of agents to one, i.e $\sum_{i=1}^{N} \mu_i = 1$. Since the mass of agents is 1, from now on all aggregate variables can also be interpeted as per capita variables. Preferences are time separable, defined over streams of consumption, given by

$$U = \sum_{t=0}^{\infty} \beta^t u\left(c_t^i\right),$$

For the purpose of this example (see Chatterjee for more general cases) assume that $u(c) = \ln(\bar{c} + c)$, with $\bar{c} + c \geq 0$ where we allow $\bar{c} \leq 0$ in order to be able to model a subsistence level for consumption.

Household's problem– We assume complete markets, so we can use the Arrow-Debreu formulation of the household problem (with the time-zero lifetime budget constraint). The maximization problem of household i can therefore be stated as (normalizing $p_0 = 1$)

$$\max_{\{c_t^i\}} \sum_{t=0}^{\infty} \beta^t u\left(c_t^i\right)$$

$$s.t.$$

$$\sum_{t=0}^{\infty} p_t c_t^i \leq a_0^i = s_0^i \sum_{t=0}^{\infty} p_t d_t \tag{1}$$

where where p_t is the price of the consumption at time t relative to consumption at time 0, a_0^i is the initial wealth of agent i in terms of time 0 consumption and d_t represent dividend paid by a representative firm (see below). Note also that, in general, we can define the wealth of agent i at time t (measured in units of time 0 consumption) as

$$p_t a_t^i = s_t^i \sum_{j=t}^{\infty} p_j d_j, \tag{2}$$

where s_t^i is the share of the firm-value owned by consumer i at time t. Indeed, by summing both sides of (2) over i and exploiting the fact that $\sum_{i=1}^{N} \mu^i s_t^i = 1$ for every t, we obtain

$$\sum_{i=1}^{N} \mu^i s_t^i p_t a_t^i = p_t a_t$$

where $p_t a_t$ is the total value of the representative firm.

Technology and firm's problem– Assume that there is a large number of firms and that each firm can operate an identical tecnology that allows to transform k units of consumption good today into $(1-\delta)k + f(k)$ units of consumption tomorrow with f strictly increasing, strictly concave and differentiable. Assume also that each firm starts out with the same amount of consumption good k_0.It can be shown that each firm will choose the same production plan (**Homework**. Show that is indeed the case. What happens if each firm starts with a different k_0? Can you still construct a representative firm? if not what else do you need to keep track to describe the evolution of the production sector?) so that we can focus on a representative firm who owns owns physical capital and makes the investment decision by solving the problem

$$\begin{aligned} a_0 &= \max_{\{k_j\}} \sum_{j=0}^{\infty} p_j \left[f(k_j) + (1-\delta) k_j - k_{j+1} \right] \\ &= \max_{\{k_j\}} \sum_{j=0}^{\infty} p_j d_j \\ d_j &\equiv \left[f(k_j) + (1-\delta) k_j - k_{j+1} \right] \\ & \quad k_0 \text{ given} \end{aligned}$$

It is also easy to see that we can write the value of the firm at an arbitrary period (in units of time 0 consumption) as

$$p_t a_t = \max_{\{k_j\}} \sum_{j=t}^{\infty} p_j \left[f(k_j) + (1-\delta) k_j - k_{j+1} \right]$$

Homework: Assume that in this economy there is a constant mass $L = 1$ of workers, that the technology is $f(K,L) = L^{1-\alpha} K^{\alpha} = K^{\alpha}$ and that in each period the representative firm pays dividend $d_t = \alpha k_t^{\alpha} + (1-\delta)k_t - k_{t+1}$ to its stockholders while it pays $(1-\alpha)k_t^{\alpha}$ to workers. Show that in this case the value of the firm $a_t = k_{t+1}.$ Is this true also in the economy in which there are no workers but a fixed factor and so $d_t = k_t^{\alpha} + (1-\delta)k_t - k_{t+1}$ (i.e. stock holders receive also the remuneration to the fixed factor)? Explain why.

Solution– From the FOC of the household problem, we have:

$$\beta^t u'\left(c_t^i\right) = \lambda^i p_t \quad \Rightarrow \quad \beta^t \left(\frac{1}{\bar{c} + c_t^i} \right) = \lambda^i p_t \quad \Rightarrow \quad c_t^i = \frac{\beta^t}{\lambda^i p_t} - \bar{c}, \tag{3}$$

where λ_i is the Lagrange multiplier on the budget constraint of household i. Substituting this FOC into the time-zero lifetime budget constraint (1), we can derive an expression for the multiplier λ^i:

$$\begin{aligned}\sum_{t=0}^{\infty} p_t \left(\frac{\beta^t}{\lambda^i p_t} - \bar{c} \right) &= a_0^i \\ \frac{1}{\lambda^i (1-\beta)} - \bar{c} \sum_{t=0}^{\infty} p_t &= a_0^i \\ \left(\frac{1}{\lambda^i} \right) &= (1-\beta)\, a_0^i + (1-\beta)\, \bar{c} \sum_{t=0}^{\infty} p_t \qquad (4)\end{aligned}$$

Let's now substitute the expression on the last line into equation (3) evaluated at time $t = 0$ (remember that $p_0 = 1$) in order to solve explicitly for c_0^i:

$$\begin{aligned} c_0^i &= \left[(1-\beta)\, a_0^i + (1-\beta)\, \bar{c} \sum_{t=0}^{\infty} p_t \right] - \bar{c} \\ &= \bar{c} \left[(1-\beta) \sum_{t=0}^{\infty} p_t - 1 \right] + (1-\beta)\, a_0^i \qquad (5) \\ &= \Theta\left(p^0, \bar{c}\right) + (1-\beta)\, a_0^i, \end{aligned}$$

where $\Theta\left(p^0, \bar{c}\right)$ denotes a function of the subsistence level and of the whole price sequence $p^0 = \{p_0, p_1, ..., p_t, ...\}$.This derivation can be easily generalized for every $t > 0$ (by using the Arrow-Debreu constraint for time t) so that

$$c_t^i = \Theta\left(p^t, \bar{c}\right) + (1-\beta)\, a_t^i, \qquad (6)$$

which shows that the optimal consumption choice at time t is an *affine function* of asset holdings at time t for each type i.

More in general, when period utility belongs to the families considered by Chatterjie, then preferences share a common property. They are *homothetic*, i.e. have linear Engel curves in wealth: any given change in wealth induces the same change in consumption, independently of the wealth level .[1]Even though we have only derived it for the log-case, it is easy to check that this representation of the consumption function holds also for the other two classes of preferences considered in the paper (power and exponential utility).

Implications for aggregate dynamics– Denote aggregate variables with capital letters. The first consequence of equation (6) is that to study the dynamics of aggregate variables in this model economy, we don't need to keep track of the distribution of wealth. From (6), we derive easily that aggregate consumption only depends on aggregate variables (prices and aggregate wealth), i.e.

$$c_t = \Theta\left(p^t, \bar{c}\right) + (1-\beta)\, a_t, \qquad (7)$$

where a_t can be clearly be expressed only as a function of the sequence of prices $\{p_t\}_{t=0}^{\infty}$ and aggregate capital stocks $\{k_t\}_{t=0}^{\infty}$. The competitive aggregate quantities and prices can therefore be recovered

[1]Technically, when $\bar{c} < 0$, preferences are quasi-homothetic because the Engel curves do not start at the origin, i.e. they are not linear but affine. However, linearity of the wealth-expansion path is not affected by the constant $\bar{c}$.

from the following standard single-agent planning problem

$$\begin{aligned}&\max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \\ &s.t. \\ &c_t + k_{t+1} \leq f\left(k_t\right) + (1-\delta) k_t \\ &k_0 \text{ given}\end{aligned} \tag{PP}$$

with standard first order condition

$$u'\left(c_t\right) = \beta u'\left(c_{t+1}\right)\left[f'\left(k_{t+1}\right) + (1-\delta)\right].$$

Note that, from (3) and the equation above, we can obtain equilibrium prices through the recursion

$$\frac{p_t}{p_{t+1}} = f'\left(k_{t+1}\right) + (1-\delta), \text{ where } p_0 \equiv 1, \quad k_0 \text{ given} \tag{8}$$

Steady-state– It is easy to show that the economy will converge to the steady-state values of capital stock satisfying $f'\left(k^*\right) = 1/\beta - (1-\delta)$. Note now that in steady-state $p_t/p_{t+1} = 1/\beta$ for all t, hence from the definition of $\Theta\left(p^t, \bar{c}\right)$ in (5) we conclude that $c^i = (1-\beta) a^i$. In other words, in steady-state, the average propensity to save out of wealth is β, independently of wealth, for every type of household.

2 Equilibrium dynamics of the wealth distribution

The results described above imply that the dynamics of the aggregate variables are not affected by the evolution of the wealth distribution, but the inverse statement is not true: in general, the evolution of the wealth distribution across households (i.e. wealth inequality) depends on the dynamics of aggregate variables (prices and quantities).

To see this, note that from the lifetime budget constraint of agent i at time t (using 2)

$$\begin{aligned} c_t^i + \sum_{j=t+1}^{\infty} p_j c_j^i &= p_t a_t^i \quad \Rightarrow \quad c_t^i + p_{t+1} a_{t+1}^i = p_t a_t^i \\ \frac{c_t^i}{a_t^i} + \frac{p_{t+1} a_{t+1}^i}{a_t^i} &= 1 \quad \Rightarrow \quad \frac{p_{t+1} a_{t+1}^i}{a_t^i} = \left(1 - \frac{c_t^i}{a_t^i}\right), \end{aligned} \tag{9}$$

which expresses the growth rate of wealth for type i as a function of her consumption-wealth ratio. Note now that, from equations (6) and (7),

$$\frac{c_t^i}{a_t^i} = \frac{\Theta\left(p^t, \bar{c}\right)}{a_t^i} + (1-\beta), \quad \text{and} \quad \frac{C_t}{A_t} = \frac{\Theta\left(p^t, \bar{c}\right)}{A_t} + (1-\beta)$$

Thus, putting together this last line and (9)

$$\Theta\left(p^t, \bar{c}\right)\left(a_t^i - A_t\right) \gtrless 0 \quad \Leftrightarrow \quad \frac{c_t^i}{a_t^i} \lessgtr \frac{C_t}{A_t} \quad \Leftrightarrow \quad \frac{a_{t+1}^i}{a_t^i} \gtrless \frac{A_{t+1}}{A_t} \tag{10}$$

In other words,

$$\Theta\left(p^t, \bar{c}\right)\left(a_t^i - A_t\right) \gtrless 0 \quad \Leftrightarrow \quad \frac{s_{t+1}^i}{s_t^i} \gtrless 1$$

which means that whether consumer's i wealth share is increasing or decreasing over time depends on the sign of the constant Θ (equal for everyone) and on her relative position in the distribution. This result leads easily to the following:

Result 1: The wealth distribution remains unchanged if either of the two conditions are satisfied:

i) The economy starts with capital stock equal to its steady state level

ii) $\bar{c} = 0$

In presence of a subsistence level and with the economy starting out of the steady state things change. For example, if $\Theta > 0$ and $a_t^i > A_t$, then consumer i wealth share will grow over time, hence the distribution will become more unequal. We now determine the sign of Θ, through:

Lemma 1.1 (Chatterjee, 1994): *The common constant term of the consumption function* $\Theta\left(p^t, \bar{c}\right)$ *is greater, equal or less than zero if and only if* $\bar{c}\left(k_t - k^*\right)$ *is greater, equal or less than zero.*

Proof: *Suppose the economy grows towards the steady-state, i.e.* $k_t < k^*$. *From equation* (8), *the sequence* $\{f'\left(k_t\right)\}$ *is decreasing and the sequence* $\{p_{t+1}/p_t\}$ *will be increasing towards* β. *Therefore,* $p_{\tau+1}/p_\tau \leq \beta$ *for all* $\tau \geq t$ *where the strict inequality holds at least for some* t. *It follows that*

$$p_\tau/p_t = \left(p_\tau/p_{\tau-1}\right)\left(p_{\tau-1}/p_{\tau-2}\right)...\left(p_{t+2}/p_{t+1}\right)\left(p_{t+1}/p_t\right) < \beta^{\tau-t}.$$

From the definition of $\Theta\left(p^t, \bar{c}\right)$ *in* (5), *use the above equation to obtain*

$$\Theta\left(p^t, \bar{c}\right) = \bar{c}\left[(1-\beta)\sum_{\tau=t}^{\infty}\left(\frac{p_\tau}{p_t}\right) - 1\right] > \bar{c}\left[(1-\beta)\sum_{\tau=t}^{\infty}\beta^{\tau-t} - 1\right] = 0$$

where the first inequality follows from $\bar{c} < 0$. **QED**

The consequences for the evolution of the wealth distribution in an economy growing towards the steady-state are easy to determine, at this point. In the presence of a subsistence level ($\bar{c} < 0$), $\Theta > 0$ in a growing economy. $\Theta > 0$ implies that the average propensity to consume (save) declines (increases) with wealth. In fact, from equation (10), it is clear that agents with wealth above average will increase their wealth even more relatively to the average. In other words:

Result 2: If $\bar{c} < 0$*: (i) the wealth distribution becomes more unequal as the economy grows towards the steady-state, as rich agents accumulate more than poor agents along the transition path, and (ii) there is no change in the ranking of households in the wealth distribution, i.e., initial conditions in ranking persist forever.*

Intuition. A brief discussion is in order on why we obtain this result.The key piece here is that complete markets implies Pareto efficiency, which implies constant ratios of marginal utilities of consumption at an given date for any two agents. What does equalization of marginal utilities implies for consumption? This clearly depends on preferences. If preferences are simple logarithmic ($\bar{c} = 0$) constant marginal utility ratio implies constant consumption ratios. Note that in the log case the consumption function is $c_{it} = (1-\beta)a_{it}$. Now if $a_{it} = s_{i0}a_t$,that is if wealth ratios between any two agents are constant then consumption ratios between any two agents will also be constant;

in other words a constant wealth distribution decentralize the efficient allocation. If $\bar{c} \neq 0$ then constant ratios of marginal utilities simply does not imply constant ratios of consumption. To see this consider the simple case of two consumers, i and j and assume that in a complete markets equilibrium this ratio is constant and equal to κ

$$\frac{u'(c_{it})}{u'(c_{jt})} = \frac{c_{jt} + \bar{c}}{c_i t + \bar{c}} = \kappa$$

dividing both numerator and denominator by c_j you get

$$\frac{1 + \bar{c}/c_{jt}}{c_{it}/c_{jt} + \bar{c}/c_{jt}} = \kappa$$

and note that (besides the degenerate case in which $\kappa = 1$ and $c_{it}/c_{jt} = 1$) the consumption ratio cannot be constant, if c_{jt} grows over time. In words with this preferences the aggregate level of resources matters for distribution. Suppose for example $\bar{c} < 0$ and that the economy grows. At low level of resources marginal utility of the poor agent (which is closer to its subsistence level) is much higher than the one of the rich agent, while at high level of resources the marginal utilities are much similar (because the subsistence level is less important for both). It is therefore efficient for the poor agent to consume (relatively) more early and (relatively) less late, so that it is efficient to have a growing path of consumption inequality, which is implemented with a path of growing wealth inequality.

Robustness– We now discuss how robust this result is to some of the key assumptions made so far in the analysis: 1) all agents have same discount factor β, 2) markets are complete, 3) absence of leisure and heterogeneity in efficiency units of labor.

- When agents have different discount factors, then none of the results hold any longer. Suppose that $\bar{c} = 0$ to simplify the analysis. Then, from (6)

  $$c_t^i = \left(1 - \beta^i\right) a_t^i,$$

  therefore the average propensity to save out of wealth is higher the more patient is the individual and from (10), wealth grows faster for the more patient individuals. In this case, to characterize aggregate dynamics one needs to know the entire distribution of wealth. Notice also that in the limit, in steady-state, the most patient type holds all the wealth, and the distribution is degenerate.

- In absence of markets (autarky), every consumer has access to her own technology, but there is no trade. Each agent will solve a standard planning problem with different initial conditions K_0^i. It is easy to see that, independently of the initial conditions, each agent will converge to the same capital stock K^*, hence in the long-run the distribution of wealth is perfectly equal. Interestingly, we conclude that a less developed financial market induces less wealth inequality, in the long-run (what about welfare?). Also, there is an important parallel with the convergence literature in growth theory: just think of consumers as countries. If every country has access to the same world technology and capital is perfectly mobile across countries, then the neoclassical growth model does *not* predict convergence anymore.[2]

- When preferences are also a function of hours worked h, and households differ by their (fixed) endowment of efficiency units and by their holdings of shares of the representative firm, aggregation can also occur, under certain restrictions on preferences, for example when

$$u(c,h) = \frac{\left(c^{\alpha}(1-h)^{1-\alpha}\right)^{1-\gamma}}{1-\gamma}. \tag{11}$$

Homework: Show that this indeed the case.

3 Indeterminacy of the wealth distribution in steady-state

One very important implication of the aggregation results is that in steady-state the wealth distribution is indeterminate. Suppose again that $\bar{c} = 0$ to simplify the analysis. From (6) and (2), the set of equations characterizing the steady-state is

$$\begin{aligned} c^i &= (1-\beta)\, a^i,\ i = 1,2,...,N \\ a^i &= s^i \frac{1}{1-\beta}\left[f(K^*) - \delta K^*\right],\ i = 1,2,...,N \\ f'(K^*) &= 1/\beta - (1-\delta), \\ \sum_{i=1}^{N} \mu^i s^i &= 1, \end{aligned}$$

We therefore have $(2N+2)$ equations and $(3N+1)$ unknowns $\left(\left\{c^i, a^i, s^i\right\}_{i=1}^{N}, K^*\right)$. In other words, the multiplicity of the steady-state wealth distributions is of order $N-1$.[3]

However, suppose we start from a given wealth distribution at time 0 when the economy has not yet reached its steady-state, then the dynamics of the model are uniquely determined by results above and the final steady-state distribution is determined as well. So, let's restate this as:

Result 3 In the steady-state of the neoclassical growth model with N agents heterogeneous in initial endowments and homothetic preferences, there is a continuum of steady-state wealth distributions, with dimension $(N-1)$. However, given an initial wealth distribution $\left\{s_0^i\right\}_{i=1}^{N}$ at $t=0$, the equilibrium wealth distribution $\left\{s_t^i\right\}_{i=1}^{N}$ in every period t is uniquely determined, and so is the final steady-state distribution.

Finally, in terms of language, this whole section shows that it important to distinguish "steady-state" from "equilibrium path". In this economy, the equilibrium path is always unique (given initial conditions), but the steady-state is not.

[2] More precisely, $f'(k^i)$ would be equalized across countries, hence countries would have the same capital stock and produce the same output. Thus, there would be convergence in GDP, but not in GNP.

[3] This means that, if $N=1$ (representative agent), the steady-state is unique. If $N=2$, there is a continuum of steady-states of dimension 1, and so on.