Unequal Growth*

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April 2019

PRELIMINARY AND INCOMPLETE

Abstract

This paper argues that changes in household income dynamics in the United States over the past 50 years can account, at the same time, for the increase in income inequality and for a significant portion of the US slowdown in aggregate growth. We first apply, using US household panel data for the period 1967-2014, a simple statistical decomposition showing that aggregate growth is the sum of average growth across households plus the covariance between income growth income and levels. The data shows that, in a statistical sense, most of the growth slowdown is accounted by a fall in the covariance between income levels and income growth. It also shows that the fall in covariance is the result of income inequality increasing, coupled with a negative correlation between income growth and income levels. Second, we develop a simple structural model of household income dynamics. We introduce changes to income dynamics that are qualitatively consistent with globalization: in recent years it is harder for any household to experience sustained income growth, but, if it does so, it grows faster than in earlier years. These changes can generate patterns of inequality, aggregate growth and co-movement between growth and levels consistent with US micro and macro data.

JEL Classification Numbers: D31, O4

Key Words: Income distribution, Gibrat's law, Inequality, aggregate growth, Pareto distribution, speed of transition.

*We thank Kjetil Storesletten for an insightful discussion, David Argente, Adriana Grasso and Egor Malkov for outstanding research assistance and seminar participants at several conferences and institutions for excellent comments. The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve System.

1 Introduction

Over the past five decades, two changes in US macroeconomic landscape stand tall: one is an increase in household income inequality, the other is a slowdown in aggregate growth. This paper argues that these two changes are connected. The foundation for the connection is that Gibrat's law does not hold for households, that is income growth is very "unequal" across the income distribution. If there are changes to fundamentals that drive a change in the income distribution, such as an increase in income inequality, these changes will also lead to changes in the aggregate growth, and possibly to a growth slowdown. Our main conclusion is that there is a unified explanation, namely a change in the underlying dynamics of household income, that can go a long way in jointly accounting for the increase in inequality and the slowdown in growth. We articulate our point in several steps.

First we present a simple statistical decomposition showing that aggregate growth can be written as the average of household/individual growth, plus the cross sectional covariance between growth and income levels. Higher covariance means high income households grow faster, and this results in faster aggregate growth. This decomposition suggests that a growth slowdown might arise either because the average of individual growth falls, or because the covariance between individual income growth and individual income level falls. We then use data from PSID over the period 1967-2014 to show that the slowdown in aggregate growth is mostly accounted (in a statistical sense) by the fall in covariance between individual growth and individual income level. In order to understand the fall in covariance, we further decompose the covariance as the product of the correlation between growth and level (a measure of rank mobility), the standard deviation of income levels (a measure of income inequality) and the standard deviation of income growth rates (a measure of income instability). The same PSID data reveals that the fall in covariance is mostly a result of income inequality increasing, coupled with a negative correlation between income growth and income levels. To be more precise the data shows that, at each point in time, low income households grow, on average, faster than high income households. The data also shows that, over time, high income households have a larger share of income and low income have a lower share (i.e. there is an increase in income inequality). Putting these two facts together implies that the US economy has, in the recent decades, more income concentrated in the hands of high income slow growers and less income in the hands of the poor fast growers. Since the high income slow growers are larger this fact leads, in an accounting sense, to a growth slowdown.

The growth accounting explored in the first part of the paper clarifies that, in a world where low income households grow faster than high income households, an increase in income inequality is associated (*ceteris paribus*) to a slowdown in aggregate growth. While useful, such an accounting involves several endogenous variables, such as the cross sectional income inequality and the expected income growth of different social groups, that are jointly determined by common fundamental features of an economy. For a structural interpretation of the observed phenomena it is therefore necessary to setup a model of the joint determination of the key variables at play. To this end the second part of the paper presents a simple framework to assess whether changes to individual earning dynamics that can produce, at the same time, a fall in aggregate growth and the patterns in income inequality, rank mobility and income instability we document in the data. The set-up consider a continuum of individuals entering the labor market with an initial constant income y_0. With a Poisson hazard, these individuals are then matched to a successful job/project, that enables their income to grow at a constant rate over time (as in Jones and Kim (2018)). When on a successful growth path, individuals might lose their job/project and fall back to y_0. We first show that this set-up can generate an income distributions and a distribution of growth over income levels that matches the cross-sectional patterns in the data qualitatively (i.e. violating Gibrat's law) and quantitatively. We then show that three key changes are necessary to match the documented changes in US micro and macro data. The first is a reduction of the probability of success, the second one is a faster growth rate of individuals/household which are on the fast growth path, and the third is a fall in the initial level of income y_0. These three changes can be interpreted as a stylized version of globalization. In a global world it might more difficult to land a successful job/project, but conditional on success, the payoffs are higher (growth is faster). The finding of the model suggest that the same structural factors that have lead to an increase in income inequality in the United States, are also at the heart of the slowdown in economic growth. In particular, our calibration suggests that in spite of the fact that the higher growth rates for individuals on the fast growth path eventually lead to an overall increase of aggregate output, as in e.g. Jovanovic (2014), the transition to this new output level is slow and involves a long period of below-average output.

Literature Review To be Completed Gabaix et al. (2016); Benhabib and Bisin (2016); Jovanovic (2014); Atkinson, Piketty, and Saez (2011); Jones and Kim (2014); Cortes, Jaimovich, and Siu (2018); Luttmer (2011); Arkolakis (2016); Lucas (2000); Olley and Pakes (1996); Chetty et al. (2014); Kopczuk, Saez, and Song (2010); Benabou (1996); Guvenen et al. (2015); Guvenen, Ozkan, and Song (2014)

2 A micro decomposition of aggregate growth

In this section we present a simple statistical decomposition that connects aggregate income growth to micro-level (household or individual) income growth, cross sectional income inequality, and the cross sectional correlation between income growth and income level. These types of decompositions have been widely used in industrial organization to connect sectoral productivity growth to productivity growth in individual firms (see, among others, Olley and Pakes 1996). We find it useful to apply this decomposition to household level data (as opposed to firms), because it connects aggregate growth with household income inequality, which has a more direct welfare content than firms income inequality.

Let y_{it} be level of income of household/individual i at time t. Let Γ_{t+T} be the economy's aggregate growth over an horizon T, which is

$$\Gamma_{t+T} = \frac{E_i(y_{it+T})}{E(y_{it})} = E\left(\frac{y_{it+T}}{y_{it}}\frac{y_{it}}{E(y_{it})}\right)$$

where $E(.)$ is the cross sectional average. Now define

$$g_{i,t+T} \equiv \frac{y_{it+T}}{y_{it}} \quad , \quad s_{i,t} \equiv \frac{y_{it}}{E(y_{it})}$$

so that $\Gamma_{t+T} = E(g_{i,t+T} \cdot s_{i,t})$ where $g_{i,t+T}$ is income growth of unit i and $s_{i,t}$ the ratio between income of unit i and average income. Then, using the definition of covariance and the fact that $E(s_{i,t}) = 1$ we get

$$\Gamma_{t+T} = cov(g_{i,t+T}, s_{i,t}) + E_i(g_{i,t+T}) \tag{1}$$

or equivalently

$$\Gamma_{t+T} = corr(g_{i,t+T}, s_{i,t})\sigma(s_{i,t})\sigma(g_{i,t+T}) + E(g_{i,t+T}) \tag{2}$$

Equation 1 suggests that what matters for aggregate growth is not only the (un-weighted) average individual growth $E(g_{i,t+T})$ but the distribution of growth opportunities, as summarized by $cov(g_{i,t+T}, s_{i,t})$. The intuition for why this is the case is straightforward: the higher the covariance, the faster higher income individuals grow; since they are high income they contribute more to aggregate growth and aggregate growth is higher. Equation 2 also suggests that $cov(g_{i,t+T}, s_{i,t})$ is linked to three cross sectional moments that have a natural economic interpretation. The first, $corr(g_{i,t+T}, s_{i,t})$, is the correlation between level and growth at the individual level. This measure captures the degree of mean reversion (or economic rank mobility) in individual income dynamics. The second, $\sigma(s_{i,t})$ is the standard deviation of $s_{i,t}$, which is essentially a measure of cross sectional income inequality. The third, $\sigma(g_{i,t+T})$, is the standard deviation of the growth rate of individual income, which is a measure of cross sectional income volatility. Equation 2 also suggests that changes in any of these three quantities will be associated, *ceteris paribus*, with changes in aggregate growth. If, for example, $corr(g_{i,t+T}, s_{i,t}) < 0$ an increase in cross sectional inequality will be associated with a reduction in $cov(g_{i,t+T}, s_{i,t})$ and thus in aggregate growth. It is useful at this point to note that this decomposition is a statistical identity, so it cannot be used to make causal inferences on growth and inequality. Nevertheless it highlights that these two statistics are connected and we find it useful to document how this connection has evolved over time. In particular the main goal of the next section is to document whether the slowdown in aggregate growth in the U.S. has been associated to a slowdown in individual growth $E(g_{i,t+T})$ or to a change in the distribution of growth opportunities.

3 A decomposition of US growth: 1967-2014

Equations 1 and 2 involve both cross sectional moments and moments related to individual income growth, so in order to bring them to the data we need panel data on household/individual income. Since our main focus is aggregate growth in the United States we also want a panel which captures well aggregate US growth. For these reasons we work with the Panel Study of Income Dynamics (PSID), which is a panel of US households, selected to be representative of the whole population, collected from 1967 to 2014. Figures 1 reports aggregate growth in per capita labor income both in the PSID and the National Income and Product Accounting (NIPA).[1] The solid lines report the actual annualized growth (computed over a 4 years horizon), while the dotted lines are polynomial trends. The figure shows that growth in PSID tracks growth in NIPA quite closely. Also, importantly for our purposes, both sources show, a marked decline in long run growth over the past 20 years, as highlighted by the trends.

Figure 2 also shows that the PSID captures well the patterns of US household income inequality, as documented in from a much larger cross sectional survey, i.e. the March Current Population Survey. The figure plots a common measure of inequality, the 90/20 ratio in household labor income, derived from the two surveys.[2] It shows that both surveys

[1] The income measure in PSID is total wage and salary income plus 50% of business income for each household in the sample, divided by the total number of persons in the sample. The income measure in NIPA is compensation of employees, wages and salaries disbursement plus 50% of proprietors income, per capita. All measures are deflated using PCE deflator. See the data appendix for more details on data construction and for similar figures for different (narrower and broader) income measures.

[2] The income measure in both PSID and CPS is total wage and salary income plus 50% of household business and farm income. Inequality measures are computed for households with heads between age 25 and

Figure 1: Growth in labor income: NIPA and PSID

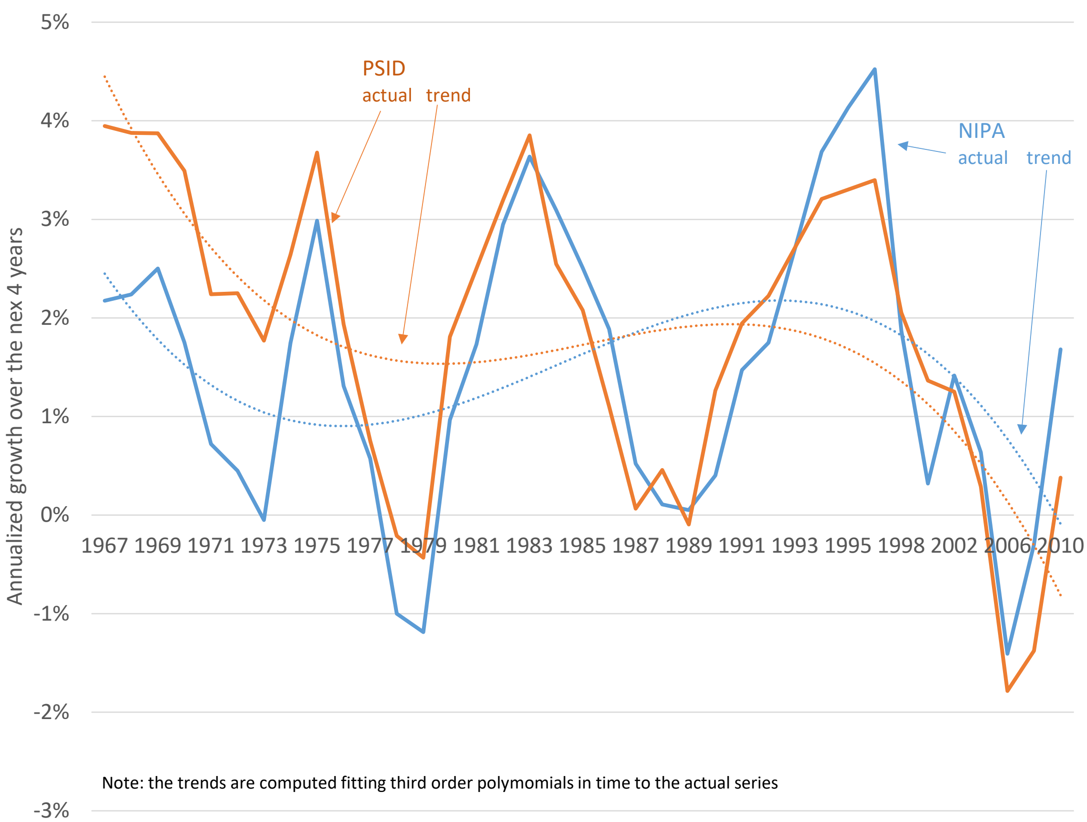
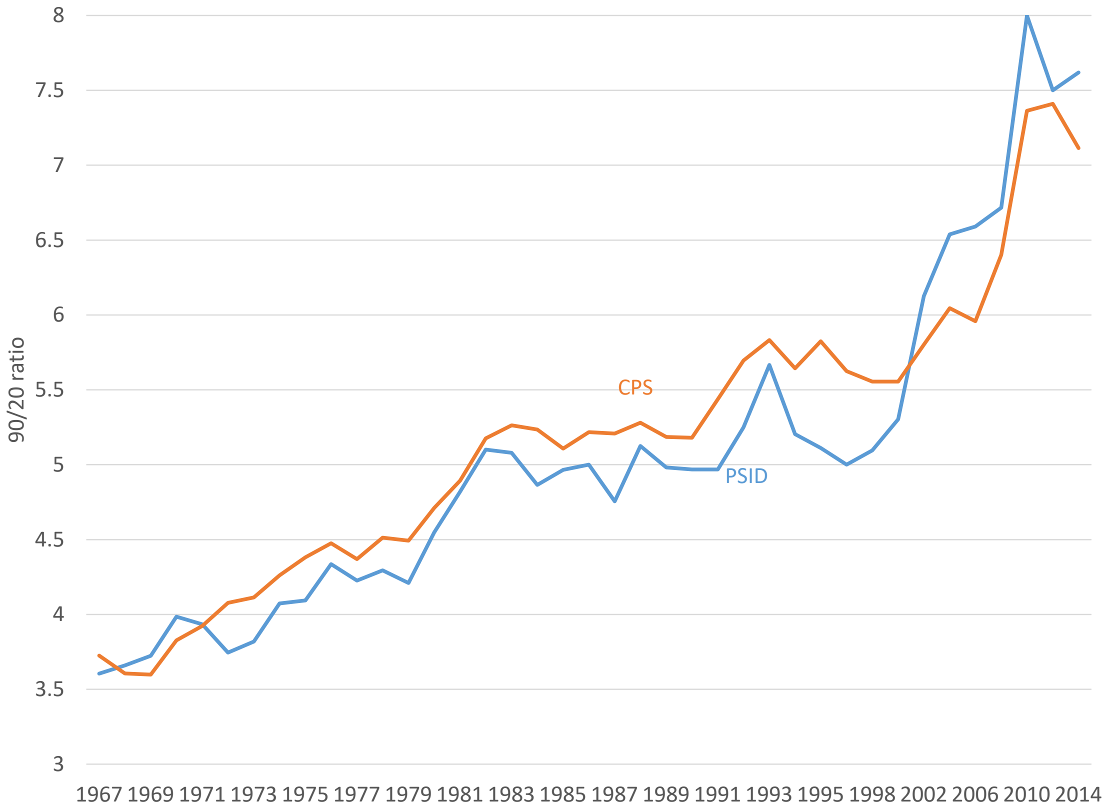


Figure 2: Inequality in labor income: PSID and CPS



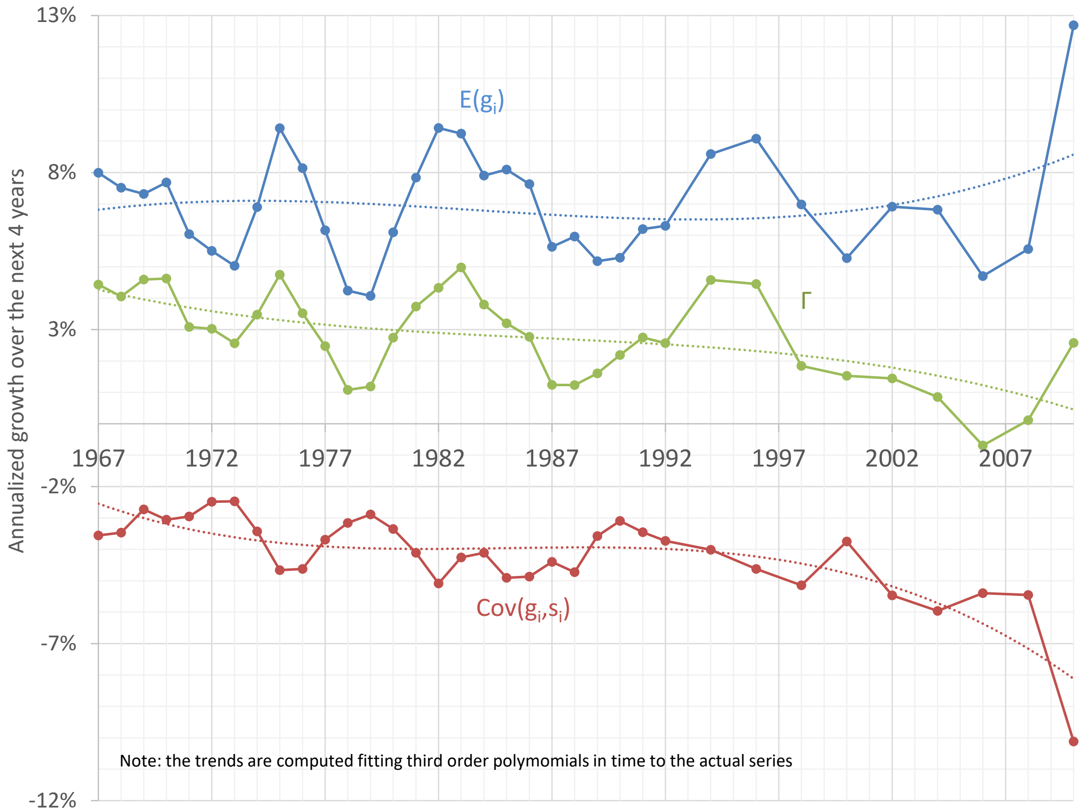
capture the well known secular increase in income inequality in the United States. Again, relevant for our purposes, it is the fact that inequality markedly increases over the past 20 years, the same exact period in which aggregate growth slows down.

Since figures 1 and 2 show that data in PSID capture well the evolution of aggregate growth and inequality, we now proceed to compute the data equivalent in PSID of equations 1 and 2. Before we do so, we briefly discuss how we select the sample for the decomposition. First we select only households with head aged between 25 and 60. The reason for this selection is that we want to exclude from our analysis changes in growth in income per capita that come, rather mechanically from changes in age composition. Second we exclude households whose income is imputed. The reason for this is that in the early 1990s PSID implemented substantial changes in the imputation procedures. Changes in the income imputation procedure in a panel data set can potentially bias measurement of income growth. Figure 3 shows the growth decomposition suggested by equation 1, where, in order to reduce noise due to measurement error in individual income, we aggregate households in 10 deciles.[3] The line

60. The average sample size in the PSID is around 4000 household per year, the size in CPS is 10 times larger.

[3]Formally let I_t by the group of households who are in the i_{th} decile of the income distribution in period

Figure 3: A decomposition of US aggregate growth



labelled Γ_{t+T} reports aggregate growth rate (annualized) over 4 years for our PSID sample. The lines labelled $E(g_{i,t+T})$ report the (unweighted) average of the growth rate across deciles in our sample, and finally the line labelled $cov(g_{i,t+T}, s_{i,t})$ reports the covariance the between the growth and the normalized level.

The upshot from 3 is that the decline in aggregate growth, displayed by the declining trend in Γ_{t+T} is mostly accounted, in a statistical sense, by the decline in covariance between growth and levels, that is $cov(g_{i,t+T}, s_{i,t})$, while the un-weighted average of growth rates in each decile, $E(g_{i,t+T})$, is, over the long run, roughly unchanged. To get some better intuition for this finding, in panel (a) of figure 4 we plot the average 4 years growth in each decile for the first four years of the PSID sample v/s the 4-years growth in the last four years of the sample for which we can compute growth.[4] The panel shows that the un-weighted average of the $g_{i,t+T}$ (i.e. the area under the two curves) is roughly constant across the two subperiods (7.6% v/s 7.4%). Perhaps surprisingly, constant average results from the bottom

t. We define $g_{i,t+T} = \frac{\sum_{i \in I_t} y_{i,t+T}}{\sum_{i \in I_t} y_{i,t}}$, that is the growth rate of income in a given decile is computed using the same group of households in t and $t+T$.

[4]The first 4 years are 1967-1970, while the last 4 years for which we can compute growth are 2004,2006,2008 and 2010. We took average over four samples to smooth out cyclical components in the growth.

deciles growing faster, and from the middle and top deciles growing slower. Panel (b) plots the share of average income in each decile (that is the s_i). The panel shows the increase inequality over the period, as reflected in the shares of the bottom deciles falling and the shares of the top deciles increasing. Finally panel (c) puts the information in the first two panels together and plots the decile growth rates, weighted by their respective shares, that is $g_{i,t+T} s_{i,t}$. The area under the lines offer a visual description of the decline in aggregate growth. Comparing growth rates (panel a) and shares (panel b) shows why growth in the later year of the sample is much weaker. The four lines in panels a and b all have the same average, yet the product of the 1967-70 lines is much higher than the product of the 2004-10 lines. This is because the covariance between the two lines has become more negative. The fast growing low deciles now have (because higher inequality) lower weight, thus dragging down growth. Similarly the slow growing high deciles now receive higher weight, similarly contributing to the reduction in growth.

So far we have established that in a statistical sense the slowdown in growth is associated to a reduction in the covariance between growth and level. Figure 5 further decomposes the trend in the covariance, using equation 2. The figure shows that the fall in the covariance is the result of two off-setting trends. On one hand the correlation between growth and levels ($corr(g_i, s_i)$) , which, in the beginning of the sample is around -0.8, becomes less negative. This would result, *ceteris paribus*, in an increase in the covariance. On the other hand the fact that income inequality ($\sigma(s_i)$) has increased, together with the fact that the correlation is negative, implies a decline in the covariance. Overall the increase in inequality dominates and thus a reduction of the covariance is observed. Nevertheless the increase (fall in absolute value), of the correlation is an important feature of the data. In particular it shows that, together with the increase in inequality, US households have experienced a substantial reduction in rank mobility, i.e. in recent years it is less likely for low income households to experience strong growth.

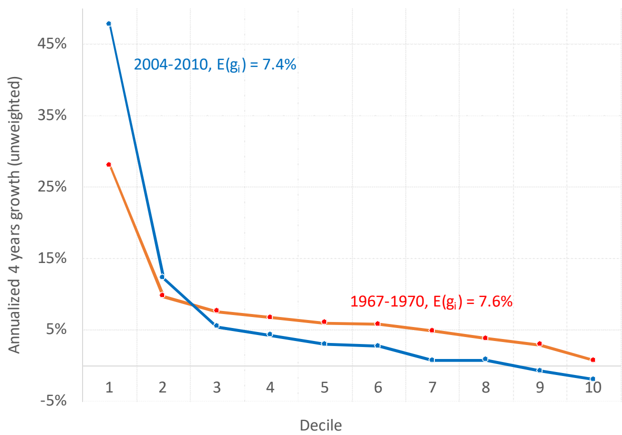
3.1 Extensions of empirical analysis

TO BE COMPLETED

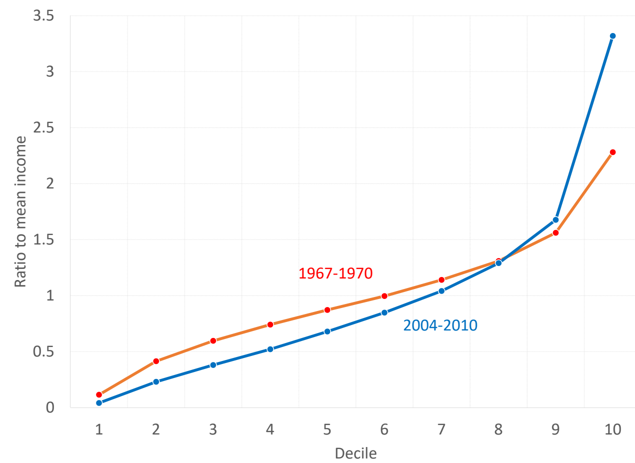
- Controlling for age, education and other demographics
- Documenting the fact on administrative data (SIPP gold standard, 1980-2012)
- Documenting the decomposition on CPS panel data (1980-2017)

Figure 4: Growth and inequality by deciles of the income distribution

(a) Growth by decile



(b) Share of income by decile



(c) Contributions to aggregate growth

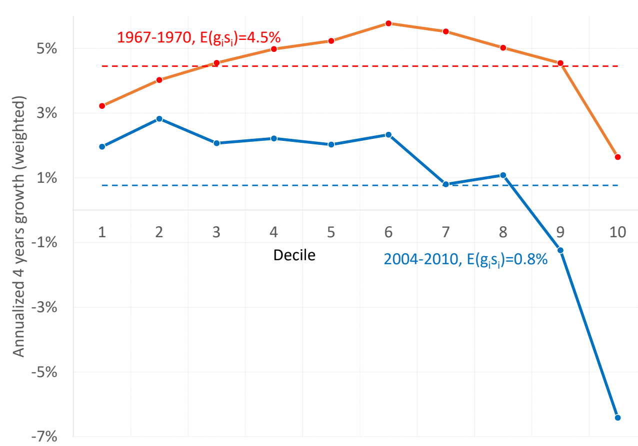
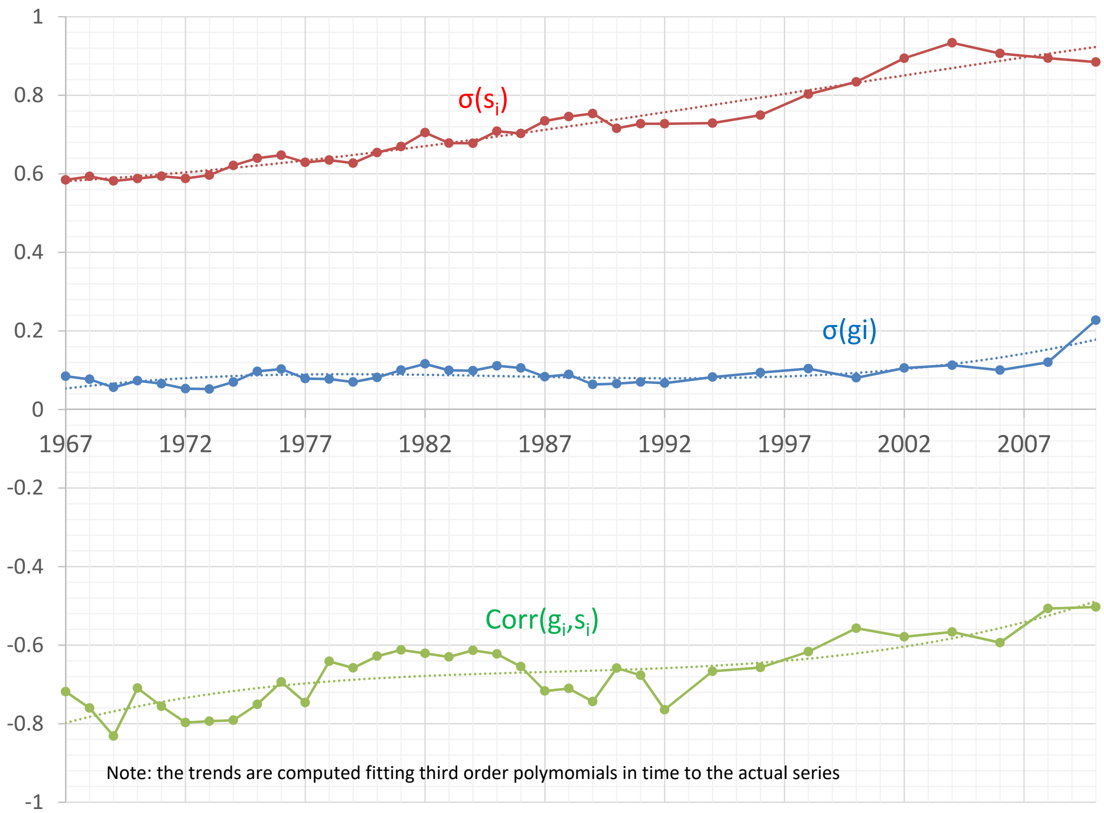


Figure 5: Decomposing the decline in covariance



4 A simple model

This section presents a simple dynamic model of the distribution of incomes. The model is deliberately simple and, in the steady-state, produces an invariant distribution of incomes that has a Pareto tail. In the steady state the model is associated with a constant aggregate income level (adding a common exogenous growth component would be trivial). We consider a once and for all income shock to some of the model fundamental parameters' that triggers a transition from the initial steady state to a new one, a thought experiment that is analogue to the one developed by Gabaix et al. (2016). The transition to the new steady state affects growth, inequality, and the other moments documented above. The main aim of the model is to analytically illustrate how a shock to the parameters of the income process will lead to a change of the cross sectional income inequality as well as to a change, during the transition, of the economy's aggregate growth rate. In spite of its mechanistic nature (we do not model any decisions) the aim of the analysis is to identify what forces may have triggered changes of the type we observed in the data, in a spirit similar to that of Lucas (2000).

The simplicity of the model has both advantages and disadvantages. The biggest advantage is that the model can be analytically solved to analyse the transition of the distribution of incomes following a fundamental shock, highlighting the various possible shocks that trigger more inequality, as well as their differential consequences in terms aggregate growth. On the other hand, its simplicity makes it hard to take it to the aggregate data without further adjustments to accommodate more sources of heterogeneity.

The setup considers a cross section of agents, each of which can be matched with a business "project" producing an income y of different types. Agents without a project have income y_0 and are matched with a new project at rate φ. Each project is destroyed at a rate δ, and as long as it remains alive the income from the project grows at rate γ so that a project surviving t periods yields the income $y(t) = y_1 e^{\gamma t}$.

In steady state the economy has a fraction $\omega \equiv \delta/(\varphi + \delta) \in (0, 1)$ of agents with no projects, i.e. income y_0. Let $\xi = \{1, 0\}$ denote the agent's state (with or without a project, respectively). Noting that the distribution of project durations is exponential with parameter δ, we compute the density of y conditional on $\xi = 1$ by a change of variables which gives

$$f(y) = \frac{\alpha\, y_1^{\alpha}}{y^{(\alpha+1)}} \quad \text{where} \quad \alpha \equiv \frac{\delta}{\gamma} \tag{3}$$

which is a Pareto distribution with CDF equal to $F(y) = 1 - \left(\frac{y_1}{y}\right)^{\alpha}$. For the distribution to have a finite mean and variance we must have $\alpha > 2$ in which case the mean income conditional on $\xi = 1$ is $\mathbb{E}\left(y \middle| \xi = 1\right) = \frac{\alpha}{\alpha-1} y_1$. Thus the mean income in the population is

$\mathbb{E}\left(y\right)=y_0\omega+\left(1-\omega\right)\frac{y_1\alpha}{\alpha-1}$.

4.1 Steady state moments

Let $s(y)\equiv y/\mathbb{E}\left(y\right)$, such that $\mathbb{E}s(y)=1$. Just like the distribution of incomes, the distribution of the $s(y)$ features a mass point ω at $s_0=y_0/\mathbb{E}\left(y\right)$ and a Pareto distribution for $s\in(s_1,\infty)$ with parameter α where $s_1=y_1/\mathbb{E}\left(y\right)$. We use the distribution $f(y)$ to compute the income levels that correspond to the deciles of the income distribution. For each decile we compute the mean income within the decile y_i, as was done in the data, and the corresponding income share $s(y_i)$. We will use the standard deviation of income shares across deciles σ_{s_i} for $i=1,2,...,10$, as our measure of income inequality.

Next we compute the expected income growth associated to an income level y over a time horizon of length T. To do so we need to compute the expected income level T periods ahead conditional on today's state $\xi=\{0,1\}$ and income level. Let $M(y,T)\equiv\mathbb{E}(y(T)|y(0)=y)$ denote the expected value of income in T periods for an agent with current income y and $\xi=1$. This is

$$M(y,T)=ye^{(\gamma-\delta)T}+\int_0^T\delta e^{-\delta s}m(T-s)ds \tag{4}$$

where $m(T)\equiv\mathbb{E}(y(T)|y(0)=y_0)$ is the expected value of income T periods from now for an agent whose current state is $\xi=0$ (i.e. in a no growth state) or

$$m(T)=y_1\int_0^T\varphi\,\theta(s)e^{(\gamma-\delta)(T-s)}ds+y_0\theta(T)\quad\text{where}\quad\theta(s)=\frac{\delta+\varphi e^{-(\varphi+\delta)s}}{\varphi+\delta} \tag{5}$$

where $\theta(s)$ is a statistic, measured at the agent level, that gives the fraction of time the agent at state $\xi=0$ will spend in that state over a time period of length s (this takes into account the possibility of leaving the state and coming back to it). Notice that the steady state fraction of agents at $\xi=0$ defined above, namely $\omega=\delta/(\delta+\varphi)$, obtains in the limit as $\lim_{s\to\infty}\theta(s)=\omega$.

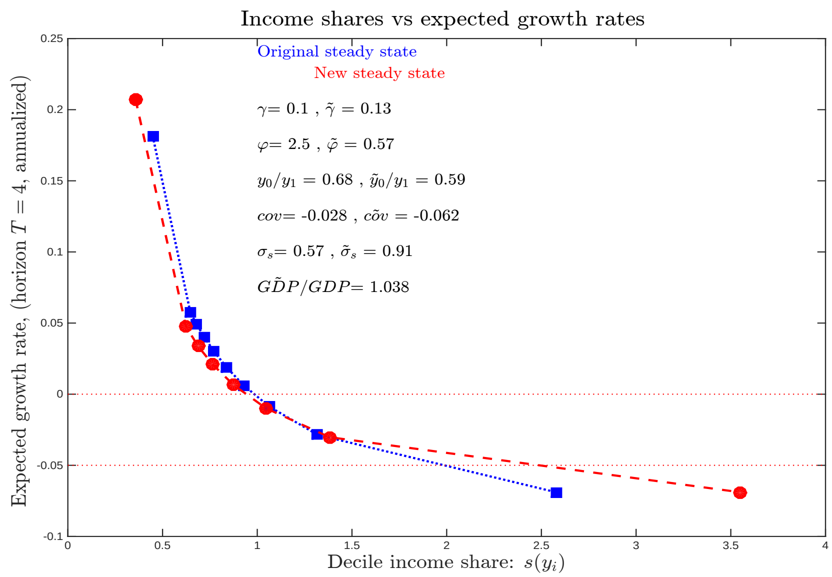
Finally, we compute the mean expected growth rate within a decile, which we denote by $\mathbb{E}(g_i|y_i,T)$. This growth rate equals $\frac{m(T)}{y_0}$ for all agents with $\xi=0$ and hence for deciles populated exclusively by such agents. For deciles populated by agents whose incomes are growing (i.e. agents with $\xi=1$), the expected growth rate is obtained by weighting the individual growth rates within the decile:

$$\mathbb{E}(g_i|y_i,T)=\int_{y\in y_i}\frac{M(y,T)}{y_i}dy\quad i=1,2,...,10$$

where the notation $y\in y_i$ is a shorthand to indicate that we are averaging across all the income levels y that belong to the interval that defines the i-th decile with mean income y_i.[5]

Following the same logic we can readily compute the higher moments, see Appendix A. These moments can be used to compute the cross sectional moments, income shares, covariances and expected growth at each decile, that correspond to the magnitudes discussed in the empirical analysis. Figure 6 illustrates this possibility for two parametrizations of the model that will be discussed in Section 6. The negative covariance between the income share and the expected growth rate, also observed in the data, obviously captures the men reversion of incomes in the model.

Figure 6: Steady-state: cross sectional patterns in the model



5 Shock and transition to a new steady state

The cross sectional patterns illustrated above are mute about the length of the transition process between the initial and the final steady state. Next we provide an analytic characterization of distribution of incomes following a once and for all shock to one or more of its fundamental parameters as a function of t, the time elapsed since the shock occurred. Solving for the time evolution of a density function is generally hard because the partial differential

[5] An obvious adjustment must be done to construct the expected growth rate for the decile that mixes agents with $\xi = 0$ and agents with $\xi = 1$.

equation involved do not admit a closed form solution. Fortunately, such difficulties can be overcome for the simple the stochastic income process considered.

Suppose at time $t = 0$ some parameters experience a once and for all change to new values $\tilde{\varphi}$ and $\tilde{\gamma}$. In particular assume that the new success rate $\tilde{\varphi}$ applies for all $t > 0$ to the pool of agents without project. The new growth rate $\tilde{\gamma}$ will only apply to successful projects initiated after $t = 0$.

The mass of agents without a project in the cross section is equal to $\omega = \delta/(\delta + \varphi)$ before the shock. After the shock, it obeys

$$\tilde{\omega}(t) = \tilde{\omega} + (\omega - \tilde{\omega})\, e^{-(\delta+\tilde{\varphi})t} \tag{6}$$

where $\tilde{\omega} = \delta/(\delta + \tilde{\varphi})$, which is the asymptotic steady state fraction of agents who are not growing.

We also allow the income for an agent without a project to evolves exogenously, t periods after the shock, as $y_0(t)$, with $y_0(0) = y_0$. We will still assume that whenever the agent exits the $\xi = 0$ state, she will start from income y_1. Thus y_1 is the (time invariant) "initial income" agents get when they start a new project. Instead, the income of the agents without the project, y_0, may change as time elapses, an assumption that we will use to analyze the consequences of the reduction of incomes in the bottom decile recorded over the past 40 years.

5.1 Income density during transition $f(y,t)$ for $\xi = 1$

Below we solve in closed form the PDE for the Kolmogorov Forward equation to compute the density of income levels during the transition. Notice that after the shock the agents with the project (i.e. with $\xi = 1$) come in 2 types. Agents with a new project, with parameter $\tilde{\gamma}$ and agents with the old project (parameter γ).

The domain for y for the new type is $y \in (y_1, y_M(t))$ with $y_M(t) = y_1 e^{\tilde{\gamma}t}$ where t is the time elapsed since the shock. Let $\tilde{f}(y,t)$ denote the density of y at time t conditional on $\xi = 1$ and the project being a new variety. We want to characterize the density $\tilde{f}(y,t)$ during a transition towards the new invariant Pareto distribution. Note that the support of this distribution is $(y_1, y_1 e^{\tilde{\gamma}t})$ where y_1 is the injection point where mass flows in at a rate $\varphi\tilde{\omega}(t)$.

The density obeys $\tilde{f}(y,t)$ the Kolmogorov forward equation

$$\frac{\partial}{\partial t}\tilde{f}(y,t) = -\frac{\partial}{\partial y}\left(\tilde{f}(y,t)\tilde{\gamma}y\right) - \delta\tilde{f}(y,t) \tag{7}$$

We use an eigenvalue-eigenfunction decomposition to solve the above PDE by separating its variables (see Appendix B for details). Conjecture $\tilde{f}(y,t) = \sum_{j=0}^{\infty} e^{\lambda_j t} f_j(y)$ then the KFE gives

$$\lambda_j f_j(y) = -f_j'(y)\tilde{\gamma}y - (\delta + \tilde{\gamma}) f_j(y) \quad \text{for} \quad j = 0, 1, 2, ...$$

So that $f_j(y) = A_j y^{-\left(1+\frac{\delta+\lambda_j}{\tilde{\gamma}}\right)}$. It is easy to show that the mass for the new type for $y \in (y_1, y_M(t))$ with $y_M(t) = y_1 e^{\tilde{\gamma}t}$ is given by $\eta(t)$ which is

$$\eta(t) = 1 - \tilde{\omega} - (\omega - \tilde{\omega})e^{-(\delta+\tilde{\varphi})t} - (1-\omega)e^{-\delta t} \tag{8}$$

which converges to $1-\tilde{\omega}$ asymptotically. We solve for λ_j, A_j by ensuring that $\int_{y_1}^{y_1 e^{\tilde{\gamma}t}} \tilde{f}(y,t)dy = \eta(t)$. This gives:

$$\tilde{f}(y,t) = (1-\tilde{\omega})\tilde{\alpha}\frac{y_1^{\tilde{\alpha}}}{y^{1+\tilde{\alpha}}} + e^{-(\delta+\tilde{\varphi})t}(\omega - \tilde{\omega})\frac{\tilde{\varphi}}{\tilde{\gamma}}\frac{y_1^{-\frac{\tilde{\varphi}}{\tilde{\gamma}}}}{y^{1-\frac{\tilde{\varphi}}{\tilde{\gamma}}}} \tag{9}$$

Notice that the transition is completely explained by just 2 eigenvalues: the one associated to the invariant distribution $\lambda_0 = 0$, and the dominant eigenvalue $\lambda_1 = -(\delta + \tilde{\varphi})$.

We can now derive an equation for the cross-sectional distribution of incomes for the agents with $\xi = 1$ ($y > y_1$), by taking into account the mass of survivors with the old project that fades out at rate δ, distributed over the domain $y \in (y_m(t), \infty)$ with $y_m(t) = y_1 e^{\gamma t}$. Hence t periods after the shock the density is

$$f(y,t) \begin{cases} \tilde{f}(y,t) & \text{for} \quad y \in (y_1, y_m(t)) \\ \\ \tilde{f}(y,t) + (1-\omega)\frac{\alpha y_1^{\alpha}}{y^{1+\alpha}} & \text{for} \quad y \in (y_m(t), y_M(t)) \\ \\ (1-\omega)\frac{\alpha y_1^{\alpha}}{y^{1+\alpha}} & \text{for} \quad y \in (y_M(t), \infty) \end{cases} \tag{10}$$

Notice that $\int_{y_1}^{\infty} f(y,t)dy = 1 - \tilde{\omega}(t) \quad \text{for all} \quad t$. See Appendix B for more information.

The above equation illustrates the convenience of our simple model. As in the seminal paper by Gabaix et al. (2016), the transition equation for the evolution of the cross sectional distribution of incomes is based on the partial differential equation that characterizes the Kolmogorov Forward equation. Two differences stand out in comparison with their analysis: first, our simpler framework allows us to derive closed form expressions for the whole distribution function during a transition, while their more general framework restricts information to the asymptotic behavior of the income distribution, namely the set of frequencies of the distribution with the slowest dynamics, as captured by the dominant eigenvalue of the transition equation. Second, in spite of its simplicity our stochastic framework mixes two fundamental sources of distribution dynamics: the evolution of the mass of agents without a project, or those with $\xi = 0$, slowly moving from ω to $\tilde{\omega}$ according to equation (6). Moreover, another source of dynamics comes from equation (10), i.e. the dynamic evolution of the incomes for agents with $\xi = 1$. The overall dynamics of the income distribution results from the combination of these forces. One consequence is that the significance of the dominant eigenvalue can be substantially muted.

5.2 Cross sectional moments during transition with $y_0(t)$

Now we compute $\mathbb{E}(g_i|y, T, t)$ the expected income growth over a time period of length T for an agent with income y, t periods after the shock. We begin by computing the expected income levels conditional on the current y in a horizon of T periods.

If the agent is in state $\xi = 0$ then the expected value of income over a time period T is computed following the logic used for the steady state in equation (5), with the difference that the formula will now use the new parameters $\tilde{\varphi}$ and $\tilde{\gamma}$ and that the income of the poor ($\xi = 0$) at time t is $y_0(t)$.

$$\tilde{m}(t, T) = y_1 \int_0^T \tilde{\varphi}\, \theta(s) e^{(\tilde{\gamma}-\delta)(T-s)} ds + \theta(T) y_0(t+T) \quad \text{where} \quad \theta(s) = \frac{\delta + \tilde{\varphi} e^{-(\tilde{\varphi}+\delta)s}}{\tilde{\varphi} + \delta} \tag{11}$$

After t periods since the shock occurred there is a mass of agents in state $\xi = 1$ which keeps growing at rate γ. The expected value of income over an horizon T for these "old" agents with current income y is

$$M^o(y, t, T) = y e^{(\gamma-\delta)T} + \int_0^T \delta e^{-\delta s} \tilde{m}(t+s, T-s) ds \tag{12}$$

Notice that this statistic depends on t, the time elapsed since the shock, since the income of the poor $y_0(t)$ depends on calendar time.

The new projects initiated after the shock grow at rate $\tilde{\gamma}$. The expected value of income for an agent with $\xi = 1$ and current income y after t periods since the shock occurred, over an horizon T, is

$$M^n(y, t, T) = y e^{(\tilde{\gamma}-\delta)T} + \int_0^T \delta e^{-\delta s} \tilde{m}(t+s, T-s) ds \tag{13}$$

The expected cross-sectional income growth t for the incomes in each decile is readily computed at each t, as was done for the steady state. As t grows these moments change because of the time varying composition of the agents, in particular the agents with the old projects growing at rate γ gradually disappear and as the new ones take over (as indicated

Table 1: Steady-state moments and model parameters

	1967-1973		2000-2010	
	Targeted moments			
	data	model	data	model
std(s_i)	0.6	0.6	0.89	0.89
cov(g_i, s_i)	-0.029	-0.029	-0.060	-0.062
std(g_i)	0.07	0.07	0.12	0.12
	calibrated model parameters			
γ		0.10		0.13
φ		2.5		0.57
y_0/y_1		0.68		0.59

Notes: Both calibrations use $y_1 = 1$ (a normalization) and $\delta = 0.25$ which implies $\omega = 0.09$ in the steady state of 1970 and $\tilde{\omega} = 0.30$ in the new steady state.

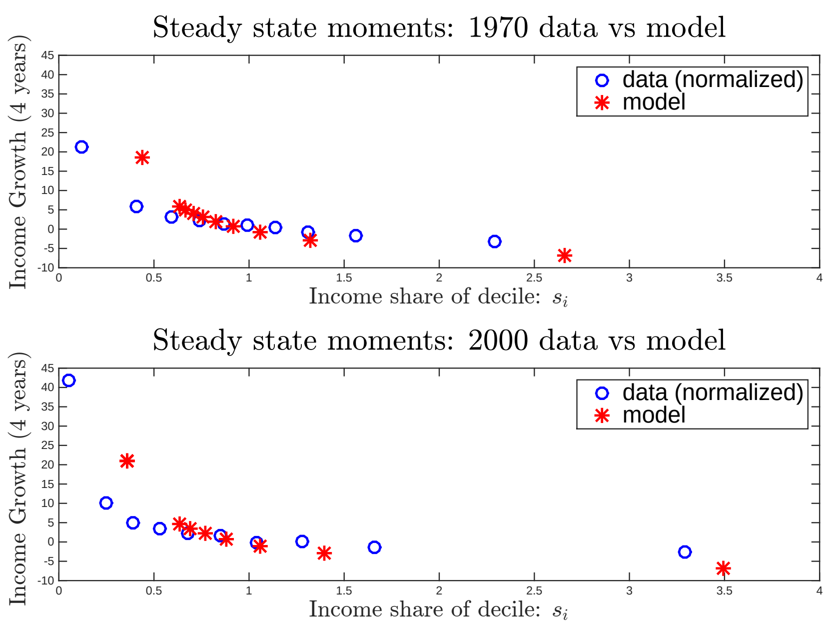
by equation (8)) and because the income of the poor changes $y_0(t)$.

6 A baseline calibration

This section presents a calibration of the model's steady state to two data points taken from the US data described above. The first data point is an measure at the beginning of the 1970s, the second datapoint is measured at the beginning of the 2000s. For each data point we consider three moments: the cross-sectional inequality, as measured by the standard deviation of the income shares s_i, by the covariance between standard deviation of the individual growth rates g_i, and by the cross-sectional covariance between g_i and s_i, as reported in Table 1. In particular for the observation of 1970 we choose the triplet of model parameters $\{\gamma, \varphi, y_0\}$, one for each observation date, under the assumption that the 1970 is a steady state. Next, we choose $\{\tilde{\gamma}, \tilde{\varphi}, \tilde{y}_0\}$ assuming that a once and for all shock occurs in 1970 that puts the economy in a transition such that, after 35 years, produces the the moment observed in the early 2000. The chosen parameters are displayed the in Table 1. In order to replicate the increase inequality of incomes and the increased (in absolute value) covariance between g_i and s_i the model implies that the growth rate $\tilde{\gamma} > \gamma$, that $\tilde{\varphi} < \varphi$ and that y_0 is a lower fraction of y_1. In other words i.e. that the shock is such that it becomes less likely for anyone to start a successful project (smaller φ), but conditional on starting a project the growth prospects are improved (higher γ).

Figure 7 displays the resulting cross sectional patterns that are implied by the chosen parameters in the steady state corresponding to each datapoint for each of the income deciles, and confronts the model predictions with the corresponding data. Figure 6 shows within a single box the patterns predicted by the model for each parametrization.

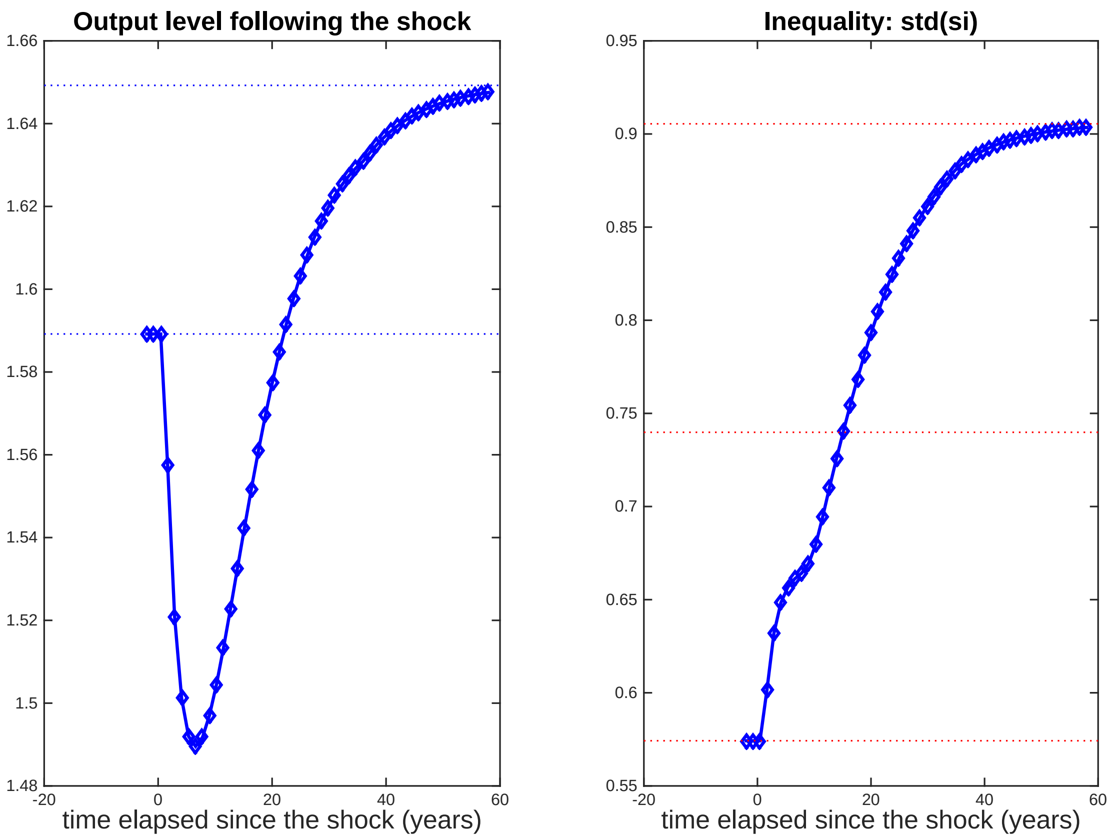
Figure 7: Cross sectional patterns in data vs model



Next we use the model to discuss the transition from the 1970s steady state to the new one. The idea, as mentioned, is to assume that in 1970 there is a once and for all change of three fundamental parameters from $\{\gamma, \varphi, y_0\}$ to $\{\tilde{\gamma}, \tilde{\varphi}, \tilde{y}_0\}$. As in the theory of section xx the change is gradual since the new parameter γ only applies to projects started after the shock and also it is assumed that the convergence from y_0 to $\tilde{y}_0$ occurs gradually at the rate ν over the 35 year period observed.

Figure 8 illustrates how this once and for all shock propagates through the economy, in terms of the aggregate output level and inequality, from the initial steady state to the new one. The transition is slow and implies an output level below the initial one for a very long period of time. in spite of the fact that the output level eventually rises, the shock causes a recession that is almost a decade long and it takes another 10 years to return to the output level recorded before the shock. In the 10 years following the shock output growth averages at about -0.5% per year, a temporary effect that is implied by the new underlying output dynamics, and which gives rise to increasing inequality, to a decreasing covariance between s_i and g_i (see and Figure 9). In spite of its simplicity, this model provides a simple mechanism that ties together more how a fundamental change of the income process may simultaneously trigger, and relate, more income inequality with less aggregate grwth, at least during the transition.

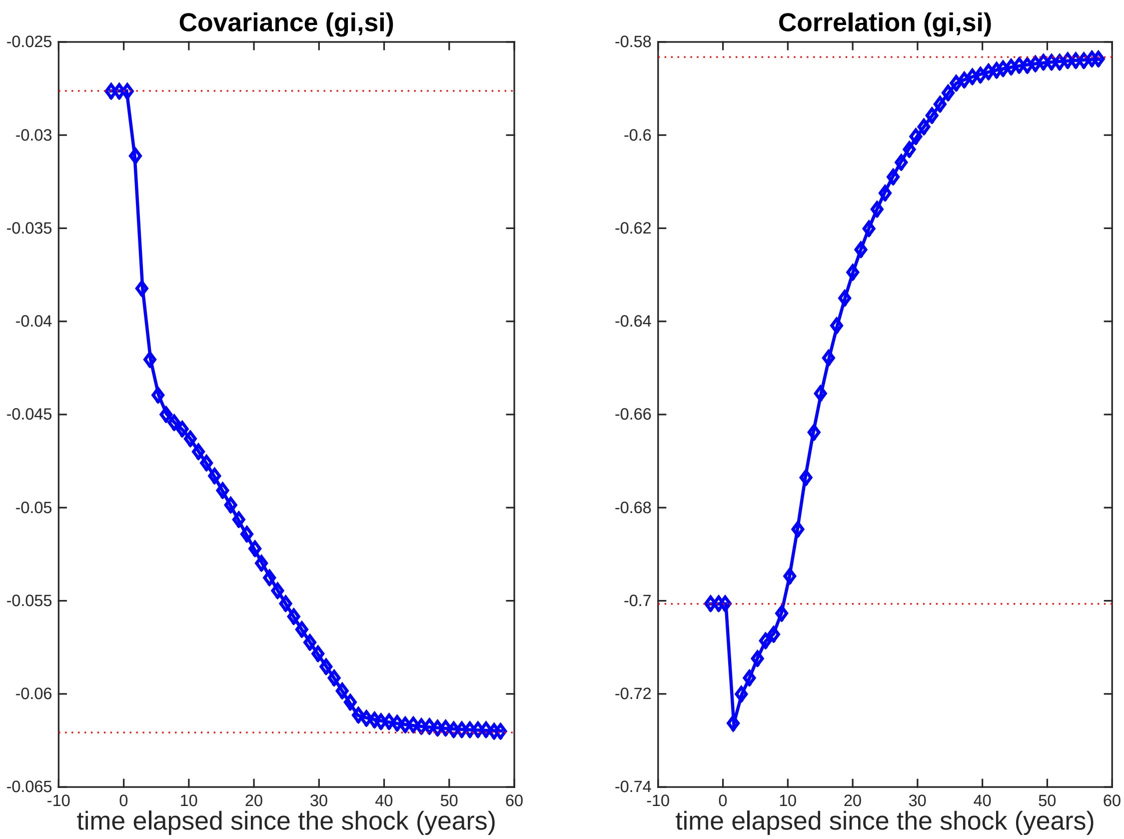
Figure 8: Dynamics: output and inequality during transition



7 Conclusion

TO BE WRITTEN

Figure 9: Dynamics: covariance and correlation during transition



A Steady state: Details on model computations

Integration of equation (5) gives

$$m(T) = y_1 \left(\frac{\alpha - \omega}{\alpha - 1} + \frac{\varphi\gamma}{\varphi + \gamma} \left(\frac{e^{-(\varphi+\delta)T}}{\varphi + \delta} - \frac{e^{-(\delta-\gamma)T}}{\delta - \gamma} \right) \right) + (y_0 - y_1) \frac{\delta + \varphi e^{-(\varphi+\delta)T}}{\varphi + \delta} \tag{14}$$

where we used $\alpha \equiv \delta/\gamma$. Notice that as $T \to \infty$ the expected income level converges to the average cross section income computed above.

Next we compute $M(y,T)$ by direct integration

$$\begin{aligned}M(y,T) =& ye^{(\gamma-\delta)T} + y_0\frac{\alpha-\omega}{\alpha-1}\left(1-e^{-\delta T}\right)\\ &+ y_0 e^{-\delta T}\left(\frac{\delta\gamma}{(\varphi+\gamma)(\varphi+\delta)}\left(1-e^{-\varphi T}\right)+\frac{\delta\varphi}{(\varphi+\gamma)(\delta-\gamma)}\left(1-e^{\gamma T}\right)\right)\end{aligned} \tag{15}$$

Thus the expected income growth in the cross section over a period of length T is

$$\mathbb{E}(g_i,T) \equiv \omega\frac{m(T)}{y_0} + (1-\omega)\int_{y_0}^{\infty} h(y)\frac{M(y,T)}{y}dy \tag{16}$$

or

$$\begin{aligned}\mathbb{E}(g_i,T) = \omega\left(\frac{\alpha-\omega}{\alpha-1}+\frac{\varphi\gamma}{\varphi+\gamma}\left(\frac{e^{-(\varphi+\delta)T}}{\varphi+\delta}-\frac{e^{-(\delta-\gamma)T}}{\delta-\gamma}\right)\right) + (1-\omega)\\ \int_{y_0}^{\infty}\alpha\frac{y_0^{\alpha}}{y^{\alpha+1}}\frac{M(y,T)}{y}dy\end{aligned} \tag{17}$$

or

$$\begin{aligned}\mathbb{E}(g_i,T) = \omega\left(\frac{\alpha-\omega}{\alpha-1}+\frac{\varphi\gamma}{\varphi+\gamma}\left(\frac{e^{-(\varphi+\delta)T}}{\varphi+\delta}-\frac{e^{-(\delta-\gamma)T}}{\delta-\gamma}\right)\right) + (1-\omega)\Big(e^{(\gamma-\delta)T}\\ + \frac{\alpha}{\alpha+1}\left(\frac{\alpha-\omega}{\alpha-1}\left(1-e^{-\delta T}\right)+e^{-\delta T}\left(\frac{\delta\gamma}{(\varphi+\gamma)(\varphi+\delta)}\left(1-e^{-\varphi T}\right)+\frac{\delta\varphi}{(\varphi+\gamma)(\delta-\gamma)}\left(1-e^{\gamma T}\right)\right)\right)\Big)\end{aligned} \tag{18}$$

Likewise let $M_2(y,T) \equiv \mathbb{E}(y^2(T)|y(0)=y)$ denote the second moment of income in T periods for an agent with current income y. This is

$$M_2(y,T) = y^2 e^{(2\gamma-\delta)T} + \int_{y_0}^{T}\delta e^{-\delta s} m_2(T-s)ds \tag{19}$$

where $m_2(T) \equiv \mathbb{E}(y^2(T)|y(0)=y_0)$ is the expected second moment of income T periods from now conditional on $\xi = 0$. Some analysis reveals that

$$m_2(T) = y_0^2\left(\int_0^T \varphi\,\theta(s)e^{(2\gamma-\delta)(T-s)}ds + \theta(T)\right) \tag{20}$$

The second moment over a period of length T is

$$\mathbb{E}(g_i^2,T) \equiv \omega\frac{m_2(T)}{y_0^2} + (1-\omega)\int_{y_0}^{\infty} h(y)\frac{M_2(y,T)}{y^2}dy \tag{21}$$

We can compute it by integrating (19) and (22).

$$m_2(T) = y_0^2 \left(\frac{\frac{\alpha}{2} - \omega}{\frac{\alpha}{2} - 1} + \frac{2\varphi\gamma}{\varphi + 2\gamma} \left(\frac{e^{-(\varphi+\delta)T}}{\varphi + \delta} - \frac{e^{-(\delta-2\gamma)T}}{\delta - 2\gamma} \right) \right) \tag{22}$$

$$\begin{aligned} M_2(y,T) =& y^2 e^{(2\gamma-\delta)T} + y_0^2 \frac{\frac{\alpha}{2} - \omega}{\frac{\alpha}{2} - 1} \left(1 - e^{-\delta T}\right) \\ &+ y_0^2 e^{-\delta T} \left(\frac{2\delta\gamma}{(\varphi + 2\gamma)(\varphi + \delta)} \left(1 - e^{-\varphi T}\right) + \frac{\delta\varphi}{(\varphi + 2\gamma)(\delta - 2\gamma)} \left(1 - e^{2\gamma T}\right) \right) \end{aligned} \tag{23}$$

therefore

$$\mathbb{E}(g_i^2, T) \equiv \omega \frac{m_2(T)}{y_0^2} + (1 - \omega) \int_{y_0}^{\infty} h(y) \frac{M_2(y,T)}{y^2} dy \tag{24}$$

$$\begin{aligned} \mathbb{E}(g_i^2, T) =& \omega \left(\frac{\frac{\alpha}{2} - \omega}{\frac{\alpha}{2} - 1} + \frac{2\varphi\gamma}{\varphi + 2\gamma} \left(\frac{e^{-(\varphi+\delta)T}}{\varphi + \delta} - \frac{e^{-(\delta-2\gamma)T}}{\delta - 2\gamma} \right) \right) + (1 - \omega) \left(e^{(2\gamma-\delta)T} \right. \\ &\left. + \frac{\frac{\alpha}{2}}{\frac{\alpha}{2} + 1} \left(\frac{\frac{\alpha}{2} - \omega}{\frac{\alpha}{2} - 1} \left(1 - e^{-\delta T}\right) + e^{-\delta T} \left(\frac{2\delta\gamma}{(\varphi + 2\gamma)(\varphi + \delta)} \left(1 - e^{-\varphi T}\right) + \frac{\delta\varphi}{(\varphi + 2\gamma)(\delta - 2\gamma)} \left(1 - e^{2\gamma T}\right) \right) \right) \right) \end{aligned} \tag{25}$$

B Details on the distribution $f(y,t)$ during a transition

Derivation of $\eta(t)$: notice that $\eta(t+\Delta) = (1-\delta\Delta)\eta(t) + \tilde{\varphi}\Delta\tilde{\omega}(t)$ which gives $\eta(t) + \Delta\eta'(t) = (1-\delta\Delta)\eta(t) + \tilde{\varphi}\Delta\tilde{\omega}(t)$ or, in the limit, $\eta'(t) = -\delta\eta(t) + \tilde{\varphi}\tilde{\omega}(t)$. This is a first order linear differential equation with solution $\eta(t) = Ce^{-\delta t} + 1 - \tilde{\omega}(t)$ where $\tilde{\omega}(t) = \frac{\delta}{\delta+\tilde{\varphi}} + \left(\frac{\delta}{\delta+\varphi} - \frac{\delta}{\delta+\tilde{\varphi}}\right) e^{-(\delta+\tilde{\varphi})t}$. By imposing the condition $\eta(0) = 0$ we find $C = -\varphi/(\delta+\varphi)$ and we have $\eta(t) = 1 - \tilde{\omega}(t) - (1-\omega)e^{-\delta t}$ which gives the equation in the text.

Solving the KFE and the associated boundary condition gives

$$\sum_{j=0}^{\infty} \frac{A_j\,\tilde{\gamma}}{\delta + \lambda_j} y_1^{-\frac{\delta+\lambda_j}{\tilde{\gamma}}} \left(e^{\lambda_j t} - e^{-\delta t}\right) = 1 - \tilde{\omega} - (\omega - \tilde{\omega})e^{-(\delta+\varphi)t} - (1-\omega)e^{-\delta t} \tag{26}$$

and matching coefficients gives the eigenvalue-eigenfunction pair associated to the invariant distribution $\lambda_0 = 0, A_0 = (1-\tilde{\omega})\tilde{\alpha}y_1^{\tilde{\alpha}}$, and another eigenvalue-eigenfunction pair associated to the transition $\lambda_1 = -(\delta+\tilde{\varphi}), A_1 = (\omega-\tilde{\omega})\frac{\tilde{\varphi}}{\tilde{\gamma}} y_1^{-\frac{\tilde{\varphi}}{\tilde{\gamma}}}$ and all others $\lambda_j = A_j = 0$ for $j = 2, 3, \ldots$, so that finally we have equation (9).

Let us compute the cumulative distribution function at time t, denoted by $F(y,t)$.

Straightforward integration gives

$$F(y,t) \begin{cases} \tilde{\omega}(t) & \text{for} \quad y \leq y_0 \\ \\ 1-(1-\tilde{\omega})\left(\frac{y_1}{y}\right)^{\tilde{\alpha}} + (\omega-\tilde{\omega})e^{-(\delta+\tilde{\varphi})t}\left(\frac{y_1}{y}\right)^{-\frac{\tilde{\varphi}}{\tilde{\gamma}}} & \text{for} \quad y \in (y_1, y_m(t)) \\ \\ 1-(1-\tilde{\omega})\left(\frac{y_1}{y}\right)^{\tilde{\alpha}} + (\omega-\tilde{\omega})e^{-(\delta+\tilde{\varphi})t}\left(\frac{y_1}{y}\right)^{-\frac{\tilde{\varphi}}{\tilde{\gamma}}} + (1-\omega)\left(e^{-\delta t} - \left(\frac{y_1}{y}\right)^{\alpha}\right) & \text{for} \quad y \in (y_m(t), y_M(t)) \\ \\ 1-(1-\omega)\left(\frac{y_1}{y}\right)^{\alpha} & \text{for} \quad y \in (y_M(t), \infty) \end{cases} \tag{27}$$

Let $F_j = \{0.1, 0.2, ..., 0.9\}$ denote the thresholds for the deciles of the distribution function at time t. The associated income thresholds from the income distribution are:

$$\begin{cases} y_0 & \text{if} \quad F_j \leq \tilde{\omega}(t) \\ F_j = 1-(1-\tilde{\omega})\left(\frac{y_1}{y_j}\right)^{\tilde{\alpha}} + (\omega-\tilde{\omega})e^{-(\delta+\tilde{\varphi})t}\left(\frac{y_1}{y_j}\right)^{-\frac{\tilde{\varphi}}{\tilde{\gamma}}} & \text{if} \quad F_j \leq F(y_m(t)) \\ F_j = 1-(1-\tilde{\omega})\left(\frac{y_1}{y_j}\right)^{\tilde{\alpha}} + (\omega-\tilde{\omega})e^{-(\delta+\tilde{\varphi})t}\left(\frac{y_1}{y_j}\right)^{-\frac{\tilde{\varphi}}{\tilde{\gamma}}} + (1-\omega)\left(e^{-\delta t} - \left(\frac{y_1}{y_j}\right)^{\alpha}\right) & \text{if} \quad F_j \leq F(y_M(t), t) \\ y_j = y_1\left(\frac{1-F_j}{1-\omega}\right)^{-\frac{1}{\alpha}} & \text{if} \quad F_j > F(y_M(t), t) \end{cases} \tag{28}$$

C Closed form expressions for cross sectional moments during the transition

We give two closed form expressions for the first and second moment of the aggregate income t periods after the shock. Equivalent magnitudes can be computed using $f(y,t)$. The aggregate income of the economy, t periods after the shock, is given by

$$\mathbb{E}(y,t) = \tilde{\omega}(t)y_0(t) + y_1\tilde{\varphi}\int_0^t \tilde{\omega}(s)e^{(\tilde{\gamma}-\delta)(t-s)}ds + (1-\omega)\int_{y_1 e^{\gamma t}}^{\infty} h(y)y dy \tag{29}$$

or

$$\mathbb{E}(y,t) = \tilde{\omega}(t)y_0(t) + y_1\left(\tilde{\varphi}\int_0^t \tilde{\omega}(s)e^{(\tilde{\gamma}-\delta)(t-s)}ds + (1-\omega)e^{(\gamma-\delta)t}\frac{\alpha}{\alpha-1}\right) \tag{30}$$

Let $\bar{y}_0 = \lim_{t\to\infty} y_0(t)$, then the output in the new steady state is

$$\lim_{t\to\infty} \mathbb{E}(y,t) = \tilde{\omega}\bar{y}_0 + \frac{\tilde{\alpha} y_1(1-\tilde{\omega})}{\tilde{\alpha}-1}$$

The second moment for aggregate income in the economy, t periods after the shock is

$$\mathbb{E}(y^2,t) = \tilde{\omega}(t)y_0^2(t) + y_1^2\left(\tilde{\varphi}\int_0^t \tilde{\omega}(s)e^{(2\tilde{\gamma}-\delta)(t-s)}ds\right) + (1-\omega)\int_{y_1e^{\gamma t}}^{\infty} h(y)y^2dy \tag{31}$$

or

$$\mathbb{E}(y^2,t) = \tilde{\omega}(t)y_0^2(t) + y_1^2\left(\tilde{\varphi}\int_0^t \tilde{\omega}(s)e^{(2\tilde{\gamma}-\delta)(t-s)}ds + \frac{(1-\omega)\alpha}{\alpha-2}e^{(2\gamma-\delta)t}\right) \tag{32}$$

Given the second moment, we compute the variance of the income shares s_i along the transition as: $\sigma^2_{s_i,t} = \frac{\mathbb{E}(y^2,t)}{(\mathbb{E}(y,t))^2} - 1$.

Integration of equation (11) gives

$$\tilde{m}(t,T) = y_1\left(\frac{\tilde{\alpha}-\tilde{\omega}}{\tilde{\alpha}-1} + \frac{\tilde{\varphi}\tilde{\gamma}}{\tilde{\varphi}+\tilde{\gamma}}\left(\frac{e^{-(\tilde{\varphi}+\delta)T}}{\tilde{\varphi}+\delta} - \frac{e^{-(\delta-\tilde{\gamma})T}}{\delta-\tilde{\gamma}}\right)\right) + (y_0(t+T)-y_1)\frac{\delta+\tilde{\varphi}e^{-(\tilde{\varphi}+\delta)T}}{\tilde{\varphi}+\delta} \tag{33}$$

where we used $\tilde{\alpha} \equiv \delta/\tilde{\gamma}$ and $\tilde{\omega} \equiv \delta/(\delta+\tilde{\varphi})$.

After t periods the expected growth rate of income over an horizon T is

$$\mathbb{E}(g_i,t,T) \equiv \tilde{\omega}(t)\frac{\tilde{m}(t,T)}{y_0(t)} + (1-\tilde{\omega}(t))\left(\int_{y_1}^{y_1e^{\tilde{\gamma}t}} h(y,\tilde{\alpha})\frac{M^n(y,t,T)}{y}dy + \int_{y_1e^{\gamma t}}^{\infty} h(y,\alpha)\frac{M^o(y,t,T)}{y}dy\right) \tag{34}$$

where $h(y,\alpha) = \alpha y_1^{\alpha}/(y^{\alpha+1})$ from equation (3).[6] Notice that $\mathbb{E}(g_i,t,T)$ is a forward looking variable (expectation over future horizons). It jumps the moment the shock hits since agents know the new parameters will apply from that moment onwards.

In the same fashion we can compute the second moment

$$\mathbb{E}(g_i^2,t,T) \equiv \tilde{\omega}(t)\frac{\tilde{m}_2(t,T)}{y_0^2(t)} + (1-\tilde{\omega}(t))\left(\int_{y_1}^{y_1e^{\tilde{\gamma}t}} h(y,\tilde{\alpha})\frac{M_2^n(y,t,T)}{y^2}dy + \int_{y_1e^{\gamma t}}^{\infty} h(y,\alpha)\frac{M_2^o(y,t,T)}{y^2}dy\right) \tag{35}$$

where

$$\tilde{m}_2(t,T) = y_1^2\int_0^T \varphi\,\theta(s)e^{(2\gamma-\delta)(T-s)}ds + \theta(T)y_0^2(t+T) \tag{36}$$

[6]Verify : note that $\forall\ t$ we have (after simple algebra)

$$1 = \tilde{\omega}(t) + (1-\tilde{\omega}(t))\left(\int_{y_1}^{y_1e^{\tilde{\gamma}t}} h(y,\tilde{\alpha})dy + \int_{y_1e^{\gamma t}}^{\infty} h(y,\alpha)dy\right)$$

$$1 = \int_{y_1}^{y_1e^{\tilde{\gamma}t}} h(y,\tilde{\alpha})dy + \int_{y_1e^{\gamma t}}^{\infty} h(y,\alpha)dy = 1 - e^{-\delta t} + e^{-\delta t}$$

which is

$$\tilde{m}_2(t,T) = y_1^2\left(\frac{\frac{\tilde{\alpha}}{2}-\tilde{\omega}}{\frac{\tilde{\alpha}}{2}-1} + \frac{2\tilde{\varphi}\tilde{\gamma}}{\tilde{\varphi}+2\tilde{\gamma}}\left(\frac{e^{-(\tilde{\varphi}+\delta)T}}{\tilde{\varphi}+\delta} - \frac{e^{-(\delta-2\tilde{\gamma})T}}{\delta-2\tilde{\gamma}}\right)\right) + \left(y_0^2(t+T) - y_1^2\right)\frac{\delta+\tilde{\varphi}e^{-(\tilde{\varphi}+\delta)T}}{\tilde{\varphi}+\delta} \tag{37}$$

and

$$M_2^o(y,t,T) = y^2 e^{(2\gamma-\delta)T} + \int_0^T \delta e^{-\delta s}\tilde{m}_2(t+s, T-s)ds \tag{38}$$

$$M_2^n(y,t,T) = y^2 e^{(2\tilde{\gamma}-\delta)T} + \int_0^T \delta e^{-\delta s}\tilde{m}_2(t+s, T-s)ds \tag{39}$$

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