Foreign Reserve Management

M. Amador\textsuperscript{1}  J. Bianchi\textsuperscript{2}  L. Bocola\textsuperscript{3}  F. Perri\textsuperscript{4}

\textsuperscript{1}Minneapolis Fed and U of Minnesota
\textsuperscript{2}Minneapolis Fed
\textsuperscript{3}Minneapolis Fed and Stanford
\textsuperscript{4}Minneapolis Fed
Over the past 20 years massive increase in foreign reserves holdings by Central Banks around the world.
Why do central banks hold foreign reserves?

1. *Precautionary motive*: reserves used as a buffer for bad shocks

2. *Exchange rate management*: reserves used to achieve a policy for nominal exchange rates
Motivation (ctd)

Why do central banks hold foreign reserves?

1. *Precautionary motive*: reserves used as a buffer for bad shocks
2. *Exchange rate management*: reserves used to achieve a policy for nominal exchange rates

How should central banks manage their portfolio?

- *Precautionary motive*: buy assets that pay in bad times
- *Exchange rate management*: lack of a theory
Motivation (ctd)

Why do central banks hold foreign reserves?

1. *Precautionary motive*: reserves used as a buffer for bad shocks

2. *Exchange rate management*: reserves used to achieve a policy for nominal exchange rates

How should central banks manage their portfolio?

- *Precautionary motive*: buy assets that pay in bad times
- *Exchange rate management*: lack of a theory

This paper: Given exchange rates and monetary policy objectives, How should a Central Bank manage its reserve portfolio?
CB has a monetary policy objective: \( \{i, e_t, e_{t+1}\} \)

Suppose that \((1 + i)\frac{e_t}{e_{t+1}} > (1 + i^*)\)  (needs limited arbitrage)
CB has a monetary policy objective: \( \{i, e_t, e_{t+1}\} \)

Suppose that \((1 + i) \frac{e_t}{e_{t+1}} > (1 + i^*)\)  (needs limited arbitrage)

- Euler equation in the domestic market

\[
u'(c_t) = \beta \left[(1 + i) \frac{e_t}{e_{t+1}}\right] u'(c_{t+1})
\]
CB has a monetary policy objective: \( \{i, e_t, e_{t+1}\} \)

Suppose that \((1 + i)\frac{e_t}{e_{t+1}} > (1 + i^*)\) (needs limited arbitrage)

- Euler equation in the domestic market
  \[
  u'(c_t) = \beta \left[ (1 + i)\frac{e_t}{e_{t+1}} \right] u'(c_{t+1})
  \]

- **Unique** consumption profile \(\Rightarrow\) Requires foreign reserve accumulation (but no portfolio choice) by the CB
Foreign reserve management without uncertainty

CB has a monetary policy objective: \( \{i, e_t, e_{t+1}\} \)

Suppose that \((1 + i) \frac{e_t}{e_{t+1}} > (1 + i^*)\) (needs limited arbitrage)

- Euler equation in the domestic market

\[
u'(c_t) = \beta \left[ (1 + i) \frac{e_t}{e_{t+1}} \right] u'(c_{t+1})
\]

- Unique consumption profile \(\Rightarrow\) Requires foreign reserve accumulation (but no portfolio choice) by the CB

Policy has two costs

- Current consumption is too low
- Resource loss, as foreigners exploit interest differential
With uncertainty, consider similar policy violating interest parity
With uncertainty, consider similar policy violating interest parity

- Euler equation:

\[ u'(c_t) = \beta \mathbb{E} \left[ (1 + i) \frac{e_t}{e_{t+1}} u'(c_{t+1}) \right] \]

- **Multiple** consumption profiles consistent with same targets
Foreign reserve management with uncertainty

With uncertainty, consider similar policy violating interest parity

- **Euler equation:**

\[ u'(c_t) = \beta \mathbb{E} \left[ (1 + i) \frac{e_t}{e_{t+1}} u'(c_{t+1}) \right] \]

- **Multiple** consumption profiles consistent with same targets

- CB can implement *any* of them by managing its foreign reserves portfolio

  - Tilts consumption towards the future, as before
  - But can also *change consumption across states*
With uncertainty (continued)

- Thus CB has more options with uncertainty

For example:

- A negative covariance between the appreciation and future marginal utility boosts $c_t$ for *same targets*:

$$u'(c_t) = \beta \mathbb{E} \left[ (1 + i) \frac{e_t}{e_{t+1}} u'(c_{t+1}) \right]$$
With uncertainty (continued)

• Thus CB has more options with uncertainty

For example:

• A negative covariance between the appreciation and future marginal utility boosts $c_t$ for same targets:

$$u'(c_t) = \beta E \left( (1 + i) \frac{e_t}{e_{t+1}} u'(c_{t+1}) \right)$$

• But other domestic asset prices are affected

$\Rightarrow$ Potentially larger resource loss: foreigners exploit the best return differential
With uncertainty (continued)

• Thus CB has more options with uncertainty

For example:

• A negative covariance between the appreciation and future marginal utility boosts $c_t$ for *same targets*:

$$u'(c_t) = \beta \mathbb{E} \left[ (1 + i) \frac{e_t}{e_{t+1}} u'(c_{t+1}) \right]$$

• But other domestic asset prices are affected

⇒ Potentially larger resource loss: *foreigners exploit the best return differential*

Trade-off: consumption smoothing vs resource losses
Resolving the trade-off

When potential capital inflows are small – resource losses are small

- Optimal to focus on consumption smoothing
- Reserve management goal: increase consumption in states where currency appreciates
Resolving the trade-off

When potential capital inflows are small – resource losses are small

- Optimal to focus on consumption smoothing
- Reserve management goal: increase consumption in states where currency appreciates

When potential capital inflows are large – resources losses are large

- Optimal to focus on minimizing resource losses
- Purchase relatively safe foreign portfolio
• Two-period model, \( t \in \{1, 2\} \)
  - Small open economy (central bank + households)
  - International Financial Market
  - Foreign Intermediaries

• Uncertainty realized at \( t = 2 \)
  - \( s \in S \equiv \{s_2, ..., s_N\}, \pi(s) \)

• One (tradable) good, law of one price, foreign price normalized to 1
Asset markets: complete but segmented

International financial markets (IFM)

- Full set of Arrow-Debreu securities in foreign currency:
  - Security $s$: 1 unit of foreign currency in state $s$, 0 otherwise
  - Price $q(s)$ in terms of foreign currency at $t = 1$

Domestic financial market

- Full set of Arrow-Debreu securities in domestic currency
  - Security $s$: 1 unit of domestic currency in state $s$, 0 otherwise
  - Price $p(s)$ in terms of domestic currency at $t = 1$

Foreign Intermediaries

- Trade securities with SOE & IFM and have limited capital
Households

- Endowment: \((y_1, \{y_2(s)\})\), transfers: \(\{T_2(s)\}\)

\[
\max_{c_1, \{c_2(s), a(s), f(s)\}} \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s) u(c_2(s)) \right\}
\]

subject to:

\[
y_1 = c_1 + \sum_{s \in S} \left[ q(s)f(s) + p(s) \frac{a(s)}{e_1} \right]
\]

\[
y_2(s) + T_2(s) + f(s) + \frac{a(s)}{e_2(s)} = c_2(s) \quad \forall s \in S
\]

\[
f(s) \geq 0, \quad \forall s \in S
\]

\(e_1, e_2(s)\): exchange rates at \(t = 1\) and \(t = 2\)

\(f(s), a(s)\): holdings of foreign and domestic security \(s\)
Households

- Endowment: \((y_1, \{y_2(s)\})\), transfers: \(\{T_2(s)\}\)

\[
\max_{c_1, \{c_2(s), a(s), f(s)\}} \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s) u(c_2(s)) \right\}
\]

subject to:

\[
y_1 = c_1 + \sum_{s \in S} \left[ q(s)f(s) + p(s) \frac{a(s)}{e_1} \right]
\]

\[
y_2(s) + T_2(s) + f(s) + \frac{a(s)}{e_2(s)} = c_2(s) \quad \forall s \in S
\]

\[f(s) \geq 0, \quad \forall s \in S\]

\[e_1, e_2(s): \text{ exchange rates at } t = 1 \text{ and } t = 2\]

\[f(s), a(s): \text{ holdings of foreign and domestic security } s\]
Foreign Intermediaries

- Endowed with capital $\bar{w}$

\[
\max_{\{d_1^*, d_2^*(s), a^*(s), f^*(s)\}} \quad d_1^* + \sum_{s \in S} \pi(s) \Lambda(s) d_2^*(s)
\]

subject to:

\[
\bar{w} = d_1^* + \sum_{s \in S} p(s) \frac{a^*(s)}{e_1} + \sum_{s \in S} q(s) f^*(s)
\]

\[
d_2^*(s) = \frac{a^*(s)}{e_2(s)} + f^*(s) \quad \forall s \in S
\]

\[
f^*(s) \geq 0 \quad a^*(s) \geq 0, \quad \forall s \in S
\]

Consider $\Lambda(s) = \frac{q(s)}{\pi(s)}$ (same SDF as IFM)
Foreign Intermediaries

- Endowed with capital $\bar{w}$

\[
\max_{\{d_1^*, d_2^*(s), a^*(s), f^*(s)\}} \quad d_1^* + \sum_{s \in S} \pi(s) \Lambda(s) d_2^*(s)
\]

subject to:

\[
\bar{w} = d_1^* + \sum_{s \in S} p(s) \frac{a^*(s)}{e_1} + \sum_{s \in S} q(s) f^*(s)
\]

\[
d_2^*(s) = \frac{a^*(s)}{e_2(s)} + f^*(s) \quad \forall s \in S
\]

\[
f^*(s) \geq 0 \quad a^*(s) \geq 0, \quad \forall s \in S
\]

Consider $\Lambda(s) = \frac{q(s)}{\pi(s)}$ (same SDF as IFM)
Central Bank

- CB has an objective for the nominal interest rate and exchange rates that we take as given: \((i, e_1, \{e_2(s)\})\)

\[
1 + i = \left( \sum_{s \in S} p(s) \right)^{-1} \quad \text{(NIRC)}
\]
Central Bank

- CB has an objective for the nominal interest rate and exchange rates that we take as given: \((i, e_1, \{e_2(s)\})\)

\[
1 + i = \left(\sum_{s \in S} p(s)\right)^{-1} \quad \text{(NIRC)}
\]

- CB achieves its objective by managing its balance sheet: invest \(\{A(s), F(s)\}\); and transfers \(\{T_2(s)\}\) to households, subject to budget constraints
Central Bank

• CB has an objective for the nominal interest rate and exchange rates that we take as given: $(i, e_1, \{e_2(s)\})$

\[
1 + i = \left( \sum_{s \in S} p(s) \right)^{-1}
\]

(NIRC)

• CB achieves its objective by managing its balance sheet: invest $\{A(s), F(s)\}$; and transfers $\{T_2(s)\}$ to households, subject to budget constraints

• Given objectives, CB chooses policies to maximize welfare
• CB has an objective for the nominal interest rate and exchange rates that we take as given: \((i, e_1, \{e_2(s)\})\)

\[
1 + i = \left( \sum_{s \in S} p(s) \right)^{-1} \tag{NIRC}
\]

• CB achieves its objective by managing its balance sheet: invest \(\{A(s), F(s)\}\); and transfers \(\{T_2(s)\}\) to households, subject to budget constraints

• Given objectives, CB chooses policies to maximize welfare

Same portfolio of securities as households (no hedging motive)
Characterizing equilibria: Arbitrage returns

- **Arbitrage return** for security $s$:

\[
\kappa(s) \equiv \frac{e_1}{e_2(s)p(s)} - 1
\]

$\kappa(s) > 0 \implies$ domestic security paying in state $s$ yields higher return

- Households: borrow up to limit in foreign currency security and invest in domestic one.
Characterizing equilibria: Arbitrage returns

- **Arbitrage return** for security $s$:

\[
\kappa(s) \equiv \frac{e_1}{e_2(s)p(s)} - 1
\]

\(\kappa(s) > 0 \Rightarrow\) domestic security paying in state $s$ yields higher return

- Households: borrow up to limit in foreign currency security and invest in domestic one.

- Intermediaries: invest all available funds in security that delivers highest return.
Characterizing equilibria: Arbitrage returns

- **Arbitrage return** for security $s$:
  
  $$\kappa(s) \equiv \frac{e_1}{e_2(s)p(s)} \left(1 - \frac{1}{q(s)}\right)$$

  $\kappa(s) > 0 \Rightarrow$ domestic security paying in state $s$ yields higher return

- Households: borrow up to limit in foreign currency security and invest in domestic one.

- Intermediaries: invest all available funds in security that delivers highest return. Let $\overline{\kappa} \equiv \max_s \{\kappa(s)\}$

  $\Rightarrow$ Profits $\overline{\kappa} \times \overline{w}$
Characterizing equilibria: Resource constraint

Profits for intermediaries are losses for the SOE

\[(y_1 - c_1) + \sum_{s \in S} q(s)[y_2(s) - c_2(s)] = \bar{\kappa}\bar{w}\]
Central bank objective and interest parity

CB objective \((i, e_1, \{e_2(s)\})\) determines the risk-adjusted return differential between the risk-free domestic bond and the foreign one

\[
\Delta(i) \equiv \mathbb{E} \left[ \Lambda(s) \left( \frac{e_1}{e_2(s)} (1 + i) - (1 + i^*) \right) \right]
\]

If \(\Delta(i) > 0\), domestic assets dominate foreign assets. The opposite happens when \(\Delta(i) < 0\).

Focus on regime in which \(\Delta(i) > 0\)

• More likely if currency expected to appreciate or safe heaven.
• Requires some securities to have \(\kappa(s) \geq \Delta(i)\).
Central bank objective and interest parity

CB objective \((i, e_1, \{e_2(s)\})\) determines the risk-adjusted return differential between the risk-free domestic bond and the foreign one

\[
\Delta(i) \equiv \mathbb{E} \left[ \Lambda(s) \left( \frac{e_1}{e_2(s)}(1 + i) - (1 + i^*) \right) \right]
\]

If \(\Delta(i) > 0\), domestic assets dominate foreign assets. The opposite happens when \(\Delta(i) < 0\).
CB objective \((i, e_1, \{e_2(s)\})\) determines the risk-adjusted return differential between the risk-free domestic bond and the foreign one

\[
\Delta(i) \equiv \mathbb{E} \left[ \Lambda(s) \left( \frac{e_1}{e_2(s)}(1 + i) - (1 + i^*) \right) \right]
\]

If \(\Delta(i) > 0\), domestic assets \textit{dominate} foreign assets. The opposite happens when \(\Delta(i) < 0\).

Focus on regime in which \(\Delta(i) > 0\)

- More likely if currency expected to appreciate or safe heaven.
- Requires some securities to have \(\kappa(s) \geq \Delta(i)\)
$\Delta(i) > 0$ implies that capital flows in & $c_1$ is low
On the Need of Central Bank Intervention

$\Delta(i) > 0$ implies that capital flows in & $c_1$ is low

From BOP equation, CB needs to buy some foreign assets

$$c_1 - y_1 = \sum_s \frac{p(s)}{e_1} a^*(s) - \sum_s q(s) [f(s) + F(s)]$$
\[ \Delta(i) > 0 \] implies that capital flows in & \( c_1 \) is low

From BOP equation, CB needs to buy \textit{some} foreign assets

\[
c_1 - y_1 = \sum_s \frac{p(s)}{e_1} a^*(s) - \sum_s q(s) [ F(s) ]
\]

Households are privately \textit{unwilling} (but able) to make these trades and \textit{unable} to undo them
On the Need of Central Bank Intervention

$\Delta(i) > 0$ implies that capital flows in & $c_1$ is low

From BOP equation, CB needs to buy some foreign assets

$$c_1 - y_1 = \sum_s \frac{p(s)}{e_1} a^*(s) - \sum_s q(s) [ F(s) ]$$

Households are privately *unwilling* (but able) to make these trades and *unable* to undo them

Which assets \{F(s)\} should CB buy?
\( \Delta(i) > 0 \) implies that capital flows in & \( c_1 \) is low

From BOP equation, CB needs to buy *some* foreign assets

\[
c_1 - y_1 = \sum_s \frac{p(s)}{e_1} a^*(s) - \sum_s q(s) \left[ F(s) \right]
\]

Households are privately *unwilling* (but able) to make these trades and *unable* to undo them

**Which assets \( \{F(s)\} \) should CB buy?**

- Potential size of capital flows is key
- Today: two cases
Optimal policy. Assume $\bar{w} = 0$ and $q(s) = \beta^*\pi(s)$

- Higher $\kappa(s)$ in states in which exchange rate appreciates

Nominal bond is too attractive $\Rightarrow$ "excessive" savings

Key idea: promise low marginal utility (i.e., high $c_2$, $\kappa$) when nominal bond pays more (i.e., $e_2$ appreciates).

NIRC binds from below:

$$1 + i^* \geq E\left(e_1 e_2(s)\right) E\left(1 + \kappa(s)\right) + \text{Cov}(e_1 e_2(s), 1 + \kappa(s))$$

To reduce average intertemporal distortion $\sim E[\kappa(s)]$, increase intratemporal distortions.
Financially closed economy

Optimal policy. Assume $\bar{w} = 0$ and $q(s) = \beta^* \pi(s)$

- Higher $\kappa(s)$ in states in which exchange rate appreciates

- Nominal bond is too attractive $\Rightarrow$ “excessive” savings

- Key idea: promise low marginal utility (i.e., high $c_2, \kappa$) when nominal bond pays more (i.e., $e_2$ appreciates).
Financially closed economy

Optimal policy. Assume $\bar{w} = 0$ and $q(s) = \beta^*\pi(s)$

- Higher $\kappa(s)$ in states in which exchange rate appreciates
- Nominal bond is too attractive $\Rightarrow$ “excessive” savings
- Key idea: promise low marginal utility (i.e., high $c_2, \kappa$) when nominal bond pays more (i.e., $e_2$ appreciates).

\[
\frac{1 + i^*}{1 + i} \geq \mathbb{E} \left( \frac{e_1}{e_2(s)} \right) \mathbb{E} \left( \frac{1}{1 + \kappa(s)} \right) + \text{Cov} \left( \frac{e_1}{e_2(s)}, \frac{1}{1 + \kappa(s)} \right)
\]

- To reduce average \textit{intertemporal} distortion $\sim \mathbb{E}[\kappa(s)]$, increase \textit{intratemporal} distortions.
Financially closed economy

Optimal policy. Assume $\bar{w} = 0$ and $q(s) = \beta^* \pi(s)$

- Higher $\kappa(s)$ in states in which exchange rate appreciates

- Nominal bond is too attractive $\Rightarrow$ “excessive” savings

- Key idea: promise low marginal utility (i.e., high $c_2, \kappa$) when nominal bond pays more (i.e., $e_2$ appreciates).

NIRC binds from below:

$$\frac{1 + i^*}{1 + i} \geq \mathbb{E} \left( \frac{e_1}{e_2(s)} \right) \mathbb{E} \left( \frac{1}{1 + \kappa(s)} \right) + \text{Cov} \left( \frac{e_1}{e_2(s)}, \frac{1}{1 + \kappa(s)} \right)$$

- To reduce average *intertemporal* distortion $\sim \mathbb{E}[\kappa(s)]$, increase *intratemporal* distortions.
From arbitrage gaps to reserves, $\kappa(s) \rightarrow F(s)$

Higher $\kappa(s)$ in states in which exchange rate appreciates, imply that CB accumulates assets that pay when exchange rate appreciates

- High $\kappa(s)$ tilts consumption towards future in that state
- CB has to buy $F(s)$ to deliver consumption goods in that state

$$F(s) = c(s) - y(s)$$
From arbitrage gaps to reserves, $\kappa(s) \rightarrow F(s)$

Higher $\kappa(s)$ in states in which exchange rate appreciates, imply that CB accumulates assets that pay when exchange rate appreciates

- High $\kappa(s)$ tilts consumption towards future in that state
- CB has to buy $F(s)$ to deliver consumption goods in that state

$$F(s) = c(s) - y(s)$$

If the exchange rate appreciates in good times and assuming that output volatility is low:

- CB buys assets that pay-off in good times
Higher $\kappa(s)$ in states in which exchange rate appreciates, imply that CB accumulates assets that pay when exchange rate appreciates

- High $\kappa(s)$ tilts consumption towards future in that state
- CB has to buy $F(s)$ to deliver consumption goods in that state

$$F(s) = c(s) - y(s)$$

If the exchange rate appreciates in good times and assuming that output volatility is low:

- CB buys assets that pay-off in good times
From arbitrage gaps to reserves, $\kappa(s) \rightarrow F(s)$

Higher $\kappa(s)$ in states in which exchange rate appreciates, imply that CB accumulates assets that pay when exchange rate appreciates

- High $\kappa(s)$ tilts consumption towards future in that state
- CB has to buy $F(s)$ to deliver consumption goods in that state

$$F(s) = c(s) - y(s)$$

If the exchange rate appreciates in good times and assuming that output volatility is low:

- CB buys assets that pay-off in good times
Recall losses: \( \max_s \{ \kappa(s) \} \bar{w} \)
Financially open economy (large $\bar{w}$)

Recall losses:

$$\max_s \{\kappa(s)\} \bar{w}$$

$$\min_{\{\kappa(s)\}_{s \in S}} \left\{ \max_s \{\kappa(s)\} \right\}$$

$$s.t. \quad 0 \leq 1 + i - \sum_s \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)}$$

(NIRC)
Financially open economy (large $\bar{\nu}$)

Recall losses: $\max_s \{\kappa(s)\} \bar{\nu}$

\[
\min_{\{\kappa(s)\}_{s \in S}} \left\{ \max_s \{\kappa(s)\} \right\}
\]

\[s.t. \quad 0 \leq 1 + i - \sum_s \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)} \quad \text{(NIRC)}\]

- Optimal policy calls for equal gaps $\kappa(s) = \kappa \ \forall s$
- only allocation in which intermediaries demand risk-free bonds
- Some leeway about CB portfolio, as long as it is relatively safe
Conclusion

- Developed a framework to analyze the reserve management problem for a CB with nominal objectives
- Uncover trade-off for reserve management, based on a risk-channel
- Show that foreign reserve management can play an important and independent role when traditional monetary policy tools are constrained or devoted to alternative objectives

- Agenda
  - Implementation with specific assets (e.g. bonds and equity)
  - Capital controls on outflows
  - Closed economy implications
The Central Bank’s problem: choose \((c_1, \{c_2(s), \kappa(s)\})\) to solve

\[
V = \max \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s) u(c_2(s)) \right\}
\]

s.t. 

\[
y_1 - c_1 - \sum_s q(s)c_2(s) = L^* (\{\kappa(s)\}, \bar{\kappa})
\]

(IRC)

\[
1 - \sum_s \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)} = i
\]

(NIRC)

\[
1 + \kappa(s) = \frac{q(s)u'_1(c_1)}{\beta \pi(s)u'(c_2(s))} \quad \forall s
\]

(\(\kappa(s)\))

\[
\max_s \kappa(s) = \bar{\kappa}
\]

(\(\bar{\kappa}\))
The Central Bank’s problem: choose \((c_1, \{c_2(s), \kappa(s)\})\) to solve

\[
V = \max \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s)u(c_2(s)) \right\}
\]

s.t. \[
y_1 - c_1 - \sum_s q(s)c_2(s) = L^*\left(\{\kappa(s)\}, \bar{\kappa}\right) \tag{IRC}
\]

\[
1 - \sum_s \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)} = i \tag{NIRC}
\]

\[
1 + \kappa(s) = \frac{q(s)u_1'(c_1)}{\beta \pi(s)u'(c_2(s))} \quad \forall s \tag{\kappa(s)}
\]

\[
\max_s \kappa(s) = \bar{\kappa} \quad \tag{\bar{\kappa}}
\]
The Central Bank’s problem: choose \((c_1, \{c_2(s), \kappa(s)\})\) to solve

\[
V = \max \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s) u(c_2(s)) \right\}
\]

s.t. \(y_1 - c_1 - \sum q(s)c_2(s) = L^\star(\{\kappa(s)\}, \bar{\kappa})\) \hspace{1cm} (IRC)

\[
1 - \sum_{s} \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)} = i
\]

\(1 + \kappa(s) = \frac{q(s)u_1'(c_1)}{\beta \pi(s)u'(c_2(s))} \quad \forall s\) \hspace{1cm} (\kappa(s))

\[
\max_{s} \kappa(s) = \bar{\kappa}
\]

\(\bar{\kappa}\)
The Central Bank’s problem: choose \((c_1, \{c_2(s), \kappa(s)\})\) to solve

\[
V = \max \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s) u(c_2(s)) \right\}
\]

s.t.

\[
y_1 - c_1 - \sum_s q(s)c_2(s) = L^*\left(\{\kappa(s)\}, \bar{\kappa}\right) \tag{IRC}
\]

\[
1 - \sum_s \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)} = i \tag{NIRC}
\]

\[
1 + \kappa(s) = \frac{q(s)u_1'(c_1)}{\beta \pi(s)u'(c_2(s))} \quad \forall s \tag{\kappa(s)}
\]

\[
\max_s \kappa(s) = \bar{\kappa} \tag{\bar{\kappa}}
\]
The Central Bank’s problem: choose \((c_1, \{c_2(s), \kappa(s)\})\) to solve

\[
V(\tilde{\kappa}) = \max \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s)u(c_2(s)) \right\}
\]

subject to

\[
y_1 - c_1 - \sum_s q(s)c_2(s) = L^*(\{\kappa(s)\}, \tilde{\kappa}) \tag{IRC}
\]

\[
1 - \sum_s \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)} = i \tag{NIRC}
\]

\[
1 + \kappa(s) = \frac{q(s)u_1'(c_1)}{\beta\pi(s)u'(c_2(s))} \quad \forall s \tag{\kappa(s)}
\]

\[
1 + \tilde{\kappa} \geq \frac{q(s)u_1'(c_1)}{\beta\pi(s)u'(c_2(s))} \quad \forall s
\]

Approach: Split problem

- Solve problem for given \(\tilde{\kappa}\).
The Central Bank’s problem: choose \((c_1, \{c_2(s), \kappa(s)\})\) to solve

\[
V(\tilde{\kappa}) = \max \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s)u(c_2(s)) \right\}
\]

\[
s.t. \quad y_1 - c_1 - \sum q(s)c_2(s) = L^*({\kappa(s)}, \tilde{\kappa}) \quad (IRC)
\]

\[
1 - \sum_s \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)} = i \quad (NIRC)
\]

\[
1 + \kappa(s) = \frac{q(s)u'_1(c_1)}{\beta \pi(s)u'(c_2(s))} \quad \forall s \quad (\kappa(s))
\]

Approach: Split problem

- Solve problem for given \(\tilde{\kappa}\). Check ignored constraints
The Central Bank’s problem: choose \((c_1, \{c_2(s), \kappa(s)\})\) to solve

\[
V(\tilde{\kappa}) = \max \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s)u(c_2(s)) \right\}
\]

\[
s.t. \quad y_1 - c_1 - \sum q(s)c_2(s) = L^*(\{\kappa(s)\}, \tilde{\kappa}) \quad (IRC)
\]

\[
1 - \sum s \frac{q(s)e_1}{(1 + \kappa(s))e_2(s)} = i \quad (NIRC)
\]

\[
1 + \kappa(s) = \frac{q(s)u'_1(c_1)}{\beta \pi(s)u'(c_2(s))} \quad \forall s \quad (\kappa(s))
\]

Approach: Split problem

- Solve problem for given \(\tilde{\kappa}\). Check ignored constraints
- Solve \(V = \max_{\tilde{\kappa}} V(\tilde{\kappa})\), \(\tilde{\kappa} = \arg\max V(\tilde{\kappa})\)
CB must open positive “gaps”

For some $s, \kappa(s) > 0$

Under $\kappa(s) \leq 0$

$$\sum_{s \in S} p(s) = \sum_{s \in S} q(s) \frac{e_1}{e_2(s)(1 + \kappa(s))} \geq \sum_{s \in S} q(s) \frac{e_1}{e_2(s)} = \frac{1 + \Delta(i)}{1 + i}$$

Since $\Delta(i) > 0$,

$$\left[ \sum_{s \in S} p(s) \right]^{-1} < (1 + i)$$

Interest rate is too low relative to NIRC.
CB must open positive “gaps”

For some \( s, \kappa(s) > 0 \)

Under \( \kappa(s) \leq 0 \)

\[
\sum_{s \in S} p(s) = \sum_{s \in S} q(s) \frac{e_1}{e_2(s)(1 + \kappa(s))} \geq \sum_{s \in S} q(s) \frac{e_1}{e_2(s)} = \frac{1 + \Delta(i)}{1 + i}
\]

Since \( \Delta(i) > 0 \),

\[
\left[ \sum_{s \in S} p(s) \right]^{-1} < (1 + i)
\]

Interest rate is too low relative to NIRC.

In fact, CB always finds optimal to set \( \kappa(s) > 0 \) for all \( s \)
CB must open positive “gaps”

For some \( s, \kappa(s) > 0 \)

Under \( \kappa(s) \leq 0 \)

\[
\sum_{s \in S} p(s) = \sum_{s \in S} q(s) \frac{e_1}{e_2(s)(1 + \kappa(s))} \geq \sum_{s \in S} q(s) \frac{e_1}{e_2(s)} = \frac{1 + \Delta(i)}{1 + i}
\]

Since \( \Delta(i) > 0 \),

\[
\left[ \sum_{s \in S} p(s) \right]^{-1} < (1 + i)
\]

Interest rate is too low relative to NIRC.

In fact, CB always finds optimal to set \( \kappa(s) > 0 \) for all \( s \).
A key condition: arbitrage return on risk-free bond

Investor with one unit of the consumption good

- Invest it in domestic risk free bond:

  Cost today: 1  Benefit tomorrow: \( \left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\} \)
A key condition: arbitrage return on risk-free bond

Investor with one unit of the consumption good

- Invest it in domestic risk free bond:

  Cost today: \( 1 \)  
  Benefit tomorrow: \( \left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\} \)

- Replicate that payoff abroad:

  Cost today:  
  Benefit tomorrow: \( \left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\} \)
A key condition: arbitrage return on risk-free bond

Investor with one unit of the consumption good

- Invest it in domestic risk free bond:

  Cost today: \( 1 \)  
  Benefit tomorrow: \( \left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\} \)

- Replicate that payoff abroad:

  Cost today: \( \sum_s q(s) \left[ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right] \)  
  Benefit tomorrow: \( \left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\} \)
A key condition: arbitrage return on risk-free bond

Investor with one unit of the consumption good

- Invest it in domestic risk free bond:

  Cost today: \( 1 \)  
  Benefit tomorrow: \( \left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\} \)

- Replicate that payoff abroad:

  Cost today: \( \sum_s q(s) \left[ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right] \)  
  Benefit tomorrow: \( \left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\} \)

- If \( \sum_s q(s) \left( \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right) \neq 1 \Rightarrow \)
A key condition: arbitrage return on risk-free bond

Investor with one unit of the consumption good

• Invest it in domestic risk free bond:

Cost today: $1$  
Benefit tomorrow: $\left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\}$

• Replicate that payoff abroad:

Cost today: $\sum_s q(s) \left[ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right]$  
Benefit tomorrow: $\left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\}$

• If $\sum_s q(s) \left( \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right) \neq 1 \Rightarrow$

  • Note $\Delta(i) > 0 \iff \sum_{s \in S} q(s)(e_1(1 + i)\frac{1}{e_2(s)}) > 1$
A key condition: arbitrage return on risk-free bond

Investor with one unit of the consumption good

• Invest it in domestic risk free bond:

Cost today: 1

Benefit tomorrow: \( \left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\} \)

• Replicate that payoff abroad:

Cost today: \( \sum_s q(s) \left[ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right] \)

Benefit tomorrow: \( \left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\} \)

• If \( \sum_s q(s) \left[ \left( \frac{e_1}{\sum_s p(s)} \right)^{(1+i)-1} \frac{1}{e_2(s)} \right] \neq 1 \)

• Note \( \Delta(i) > 0 \iff \sum_{s \in S} q(s) \left( e_1(1 + i) \frac{1}{e_2(s)} \right) > 1 \)
Investor with one unit of the consumption good

- Invest it in domestic risk free bond:

  Cost today: \( 1 \)       Benefit tomorrow: \( \left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\} \)

- Replicate that payoff abroad:

  Cost today: \( \sum_s q(s) \left[ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right] \)       Benefit tomorrow: \( \left\{ \left( \frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\} \)

- If \( \sum_s q(s) \left( \left( \frac{e_1}{\sum_s p(s)^{(1+i)-1}} \right) \frac{1}{e_2(s)} \right) \neq 1 \)

- Note \( \Delta(i) > 0 \iff \sum_{s \in S} q(s)(e_1(1+i)\frac{1}{e_2(s)}) > 1 \)
Trade deficits and net foreign assets:

\[ c_1 - y_1 = \sum_s p(s) a^*(s) \frac{e_1}{e_1} - \sum_s q(s) [f(s) + F(s)] \]
Equilibrium Definition

Take a given \((i, e_1, \{e_2(s)\})\)

**Equilibrium**

HH’s consumption, \((c_1, \{c_2(s)\})\), and asset positions, \((\{a(s), f(s)\})\); Intermediaries consumption, \(\{d_1^*, d_2^*(s)\}\), and asset positions \(\{a^*(s), f^*(s)\}\); central bank transfers \(\{T_2(s)\}\), asset and liabilities \(\{A(s), F(s)\}\) ; and domestic asset prices \(\{p(s)\}\), such that:

1. HH and Intermediaries maximize taking prices as given,
2. the central bank budget constraint holds, and
3. the domestic financial markets clear:

\[ a(s) + a^*(s) + A(s) = 0 \quad \forall s \in S \]