Foreign Reserve Management

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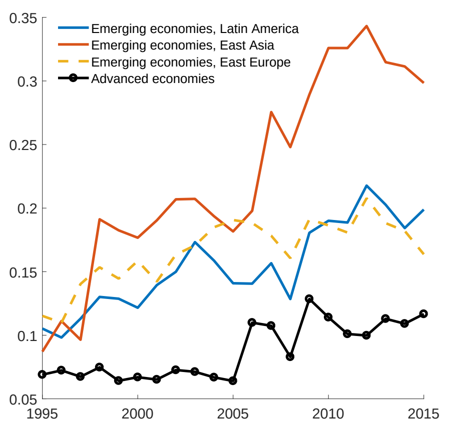
Motivation

Over the past 20 years massive increase in foreign reserves holdings by Central Banks around the world



Reserves-GDP

Motivation (ctd)

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This paper: Given exchange rates and monetary policy objectives, How should a Central Bank manage its reserve portfolio?

Foreign reserve management without uncertainty

CB has a monetary policy objective: $\{i, e_t, e_{t+1}\}$

Suppose that $(1+i)\frac{e_t}{e_{t+1}} > (1+i^{\star})$ (needs limited arbitrage)

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$$u'(c_t) = \beta \left[(1+i)\frac{e_t}{e_{t+1}} \right] u'(c_{t+1})$$

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Policy has two costs

- Current consumption is too low
- Resource loss, as foreigners exploit interest differential

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- **Multiple** consumption profiles consistent with same targets
- CB can implement *any* of them by managing its foreign reserves portfolio
    - Tilts consumption towards the future, as before
    - But can also *change consumption across states*

With uncertainty (continued)

- Thus CB has more options with uncertainty

For example:

- A negative covariance between the appreciation and future marginal utility boosts c_t for *same targets*:

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Trade-off: consumption smoothing vs resource losses

Resolving the trade-off

When potential capital inflows are small – resource losses are small

- Optimal to focus on consumption smoothing
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When potential capital inflows are large – resources losses are large

- Optimal to focus on minimizing resource losses
- Purchase relatively safe foreign portfolio

Framework

- Two-period model, $t \in \{1, 2\}$
  - Small open economy (central bank + households)
  - International Financial Market
  - Foreign Intermediaries
- Uncertainty realized at $t = 2$
  - $s \in S \equiv \{s_2, ..., s_N\}, \pi(s)$
- One (tradable) good, law of one price, foreign price normalized to 1

Asset markets: complete but segmented

International financial markets (IFM)

- Full set of Arrow-Debreu securities in foreign currency:
  - Security s: 1 unit of foreign currency in state s, 0 otherwise
  - Price $q(s)$ in terms of foreign currency at $t = 1$

Domestic financial market

- Full set of Arrow-Debreu securities in domestic currency
  - Security s: 1 unit of domestic currency in state s, 0 otherwise
  - Price $p(s)$ in terms of domestic currency at $t = 1$

Foreign Intermediaries

- Trade securities with SOE & IFM and have limited capital

Households

- Endowment: $(y_1, \{y_2(s)\})$, transfers: $(\{T_2(s)\})$

$$\max_{c_1, \{c_2(s), a(s), f(s)\}} \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s) u(c_2(s)) \right\}$$

subject to:

$$y_1 = c_1 + \sum_{s \in S} \left[q(s) f(s) + p(s) \frac{a(s)}{e_1} \right]$$

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$$f(s) \geq 0, \quad \forall s \in S$$

e_1, $e_2(s)$: exchange rates at $t = 1$ and $t = 2$

$f(s), a(s)$: holdings of foreign and domestic security s

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Foreign Intermediaries

- Endowed with capital $\bar{w}$

$$\max_{\{d_1^\star, d_2^\star(s), a^\star(s), f^\star(s)\}} d_1^\star + \sum_{s \in S} \pi(s)\Lambda(s) d_2^\star(s)$$

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- CB has an objective for the nominal interest rate and exchange rates that we take as given: $(i, e_1, \{e_2(s)\})$

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Same portfolio of securities as households (no hedging motive)

Characterizing equilibria: Arbitrage returns

- **Arbitrage return** for security s:

$$\kappa(s) \equiv \frac{\frac{e_1}{e_2(s)p(s)}}{\frac{1}{q(s)}} - 1$$

$\kappa(s) > 0 \Rightarrow$ domestic security paying in state s yields higher return

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- Intermediaries: invest all available funds in security that delivers highest return. Let $\bar{\kappa} \equiv \max_s\{\kappa(s)\}$

  $\Rightarrow$ Profits $\bar{\kappa} \times \bar{w}$

Characterizing equilibria: Resource constraint

Profits for intermediaries are losses for the SOE

$$(y_1 - c_1) + \sum_{s \in S} q(s)[y_2(s) - c_2(s)] = \bar{\kappa}\bar{w}$$

Central bank objective and interest parity

CB objective $(i, e_1, \{e_2(s)\})$ determines the risk-adjusted return differential between the risk-free domestic bond and the foreign one

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Focus on regime in which $\Delta(i) > 0$

- More likely if currency expected to appreciate or safe heaven.
- Requires some securities to have $\kappa(s) \geq \Delta(i)$

On the Need of Central Bank Intervention

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- Potential size of capital flows is key
- Today: two cases

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Optimal policy. Assume $\bar{w} = 0$ and $q(s) = \beta^{\star}\pi(s)$

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- Key idea: promise low marginal utility (i.e., high c_2, κ) when nominal bond pays more (i.e., e_2 appreciates).

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NIRC binds from below:

$$\frac{1+i^\star}{1+i} \geq \mathbb{E}\left(\frac{e_1}{e_2(s)}\right)\mathbb{E}\left(\frac{1}{1+\kappa(s)}\right) + \mathrm{Cov}\left(\frac{e_1}{e_2(s)}, \frac{1}{1+\kappa(s)}\right)$$

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From arbitrage gaps to reserves, $\kappa(s) \to F(s)$

Higher $\kappa(s)$ in states in which exchange rate appreciates, imply that CB accumulates assets that pay when exchange rate appreciates

- High $\kappa(s)$ tilts consumption towards future in that state
- CB has to buy $F(s)$ to deliver consumption goods in that state

$$F(s) = c(s) - y(s)$$

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- Optimal policy calls for equal gaps $\kappa(s) = \kappa\ \forall s$
  - only allocation in which intermediaries demand risk-free bonds
- Some leeway about CB portfolio, as long as it is relatively safe

Details

Conclusion

- Developed a framework to analyze the reserve management problem for a CB with nominal objectives
- Uncover trade-off for reserve management, based on a risk-channel
- Show that foreign reserve management can play an important and independent role when traditional monetary policy tools are constrained or devoted to alternative objectives
- Agenda
    - Implementation with specific assets (e.g. bonds and equity)
    - Capital controls on outflows
    - Closed economy implications

The Central Bank's problem: choose $(c_1, \{c_2(s), \kappa(s)\})$ to solve

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V \quad &= \max \left\{ u(c_1) + \beta \sum_{s \in S} \pi(s) u(c_2(s)) \right\} \\
s.t. \quad y_1 - c_1 - \sum q(s) c_2(s) &= L^\star(\{\kappa(s)\}, \bar{\kappa})) && \text{(IRC)} \\
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Approach: Split problem

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$$1 - \sum_s \frac{q(s) e_1}{(1 + \kappa(s)) e_2(s)} = i \qquad \text{(NIRC)}$$

$$1 + \kappa(s) = \frac{q(s) u_1'(c_1)}{\beta \pi(s) u'(c_2(s))} \ \forall s \qquad (\kappa(s))$$

Approach: Split problem

- Solve problem for given $\tilde{\kappa}$. Check ignored constraints
- Solve $V = \max_{\tilde{\kappa}} V(\tilde{\kappa}), \quad \bar{\kappa} = \operatorname{argmax} V(\tilde{\kappa})$

Back

CB must open positive "gaps"

For some $s, \kappa(s) > 0$

Under $\kappa(s) \leq 0$

$$\sum_{s\in S} p(s) = \sum_{s\in S} q(s)\frac{e_1}{e_2(s)(1+\kappa(s))} \geq \sum_{s\in S} q(s)\frac{e_1}{e_2(s)} = \frac{1+\Delta(i)}{1+i}$$

Since $\Delta(i) > 0$,

$$\left[\sum_{s\in S} p(s)\right]^{-1} < (1+i)$$

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A key condition: arbitrage return on risk-free bond

Investor with one unit of the consumption good

- Invest it in domestic risk free bond:

Cost today: 1 Benefit tomorrow: $\left\{ \left(\frac{e_1}{\sum_s p(s)} \right) \frac{1}{e_2(s)} \right\}$

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    - Note $\Delta(i) > 0 \iff \sum_{s \in S} q(s)(e_1(1+i)\frac{1}{e_2(s)}) > 1$

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Characterizing equilibria: Balance of Payment

- Trade deficits and net foreign assets:

$$\underbrace{c_1 - y_1}_{\text{trade deficit}} = \underbrace{\frac{\sum_s p(s)a^\star(s)}{e_1}}_{\text{foreign liabilities}} - \underbrace{\sum_s q(s)\left[f(s) + F(s)\right]}_{\text{foreign assets}}$$

Equilibrium Definition

Take a given $(i, e_1, \{e_2(s)\})$

Equilibrium

HH's consumption, $(c_1, \{c_2(s)\})$, and asset positions, $(\{a(s), f(s)\})$; Intermediaries consumption, $\{d_1^\star, d_2^\star(s)\}$, and asset positions $(\{a^\star(s), f^\star(s)\})$; central bank transfers $(\{T_2(s)\})$, asset and liabilities $(\{A(s), F(s)\})$; and domestic asset prices $\{p(s)\}$, such that:

1. HH and Intermediaries maximize taking prices as given,
2. the central bank budget constraint holds, and
3. the domestic financial markets clear:

$$a(s) + a^\star(s) + A(s) = 0 \quad \forall s \in S$$