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Inflation, Debt, and Default[1]

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Abstract

We study how the co-movement of inflation and the business cycle affects real interest rates and the likelihood of debt crises. First we show that for advanced economies periods in which the co-movement between inflation and the business cycle is positive, real interest rates tend to be low. A positive co-movement of inflation with the cycle raises the returns of nominal bonds in bad times, making them a good hedge against aggregate risk. However, such pro-cyclicality also generates default risk, since nominal debt becomes more expensive for the government when the economy deteriorates. In order to evaluate both effects we develop a model of sovereign default on domestic nominal debt with exogenous inflation risk and domestic risk averse agents. Counter-cyclical inflation tend to be *substitute* with default, while pro-cyclical inflation is *complement* with it. In good times, when default is not an issue, pro-cyclical inflation yields lower interest rates. In bad times, when default is possible, pro-cyclical inflation can trigger increase in default risk and spikes in real interest rates.

KEYWORDS: inflation risk, domestic nominal debt, interest rates, sovereign default
JEL CLASSIFICATION CODES: E31, F34, G12, H63

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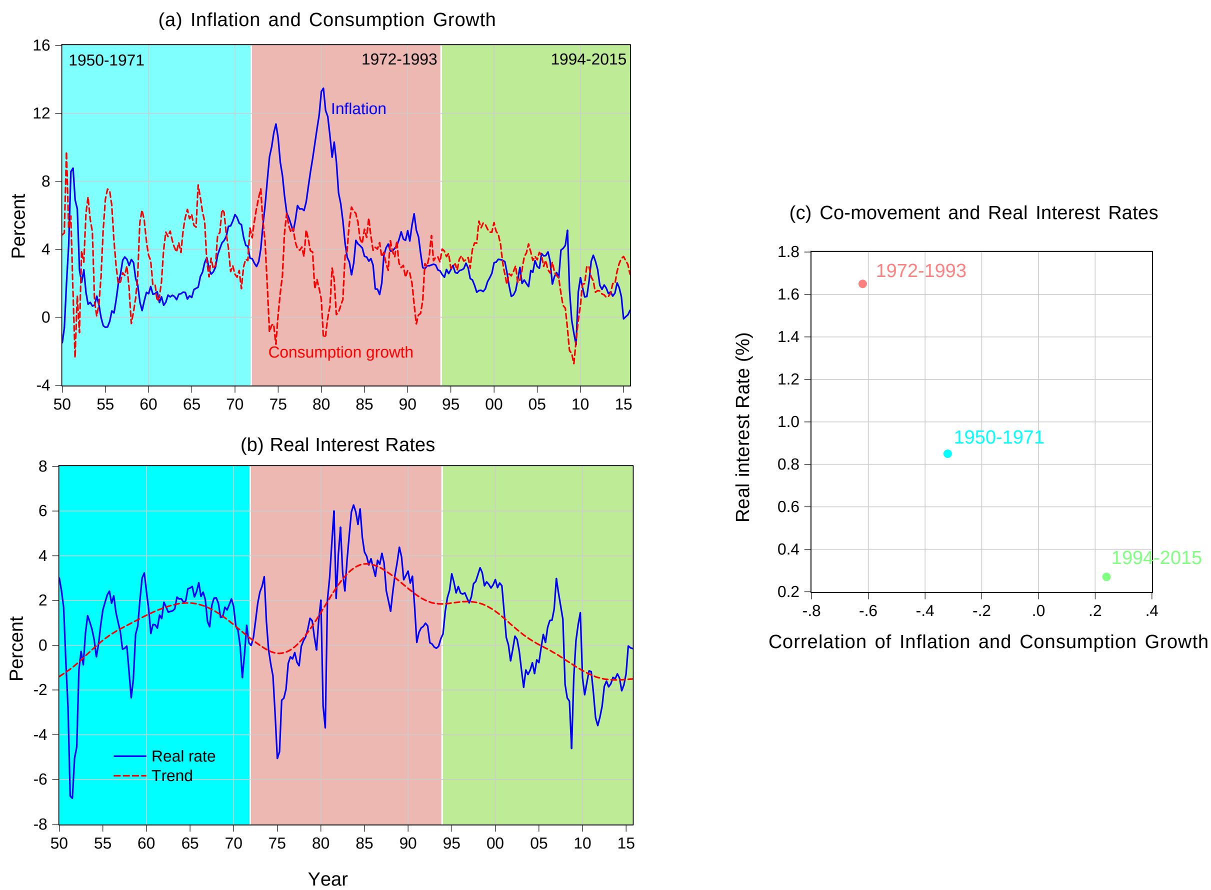
1 Introduction

Over the past 50 years inflation dynamics over the business cycle in advanced economies has changed quite dramatically, as we have observed periods of highly counter-cyclical inflation, as well as periods of pro-cyclical inflation. The goal of this paper is to study how changes in the co-movement between inflation and economic activity affect real sovereign yields, debt dynamics, and debt crises. To be more concrete, in Figure 1 we provide some motivating evidence for the mechanism we want to highlight.

Figure 1: Inflation and Consumption Growth and Real Rates in the U.S.



Note: Inflation is the log difference between CPI in quarter t and t-4. Consumption growth is the log difference in real personal consumption expenditures over the same interval. Real interest rates are nominal rates on government securities (from the IMF IFS database) minus expected inflation computed using a linear univariate forecasting model estimated on actual inflation.

In panel (a), we plot quarterly time series for year-on-year U.S. inflation and consumption growth from 1950 to 2015. The point of the panel is to highlight changes in the co-movement of two series over three equal length sub-samples. It shows how in the first sub-sample (1950–1971), the co-movement between inflation and consumption growth is mildly negative, turns to strongly negative in the second sub-sample (1972–1993), and finally becomes positive in the most recent sample (1994–2015). If inflation co-varies positively with domestic consumption growth, then returns on domestic nominal debt are high (low) when consumption growth is low (high). This feature makes domestic nominal bonds less risky from a domestic investor's perspective, and thus — if government debt is mostly held domestically, as it is in most developed countries — they should trade, *ceteris paribus*, at a lower real interest rate.[2] The second and third panels in Figure 1 show that this indeed is the case. Panel (b) plots the U.S. real interest rate (along with its trend depicted by the dashed line) over the same sample, while panel (c) plots the average real rate and the average co-movement between inflation and consumption growth in each of the three sub-samples. Notice how the middle sample, which displays the most negative inflation consumption growth co-movement, also is the one with the highest real rate. The most recent sample — where the co-movement has turned positive — displays the lowest real rate, while the early sample has intermediate co-movement and an intermediate real rate.

Altogether, this evidence in Figure 1 suggests that the co-movement between inflation and consumption growth is closely connected with the real yield on government debt. The evidence, though striking, is obviously not conclusive as there might be a variety of other factors inducing this pattern. For this reason, in the first part of this paper, we establish this relationship in a more systematic fashion. In particular, we show that for a large sample of advanced economies, in countries and periods in which the co-movement of inflation with domestic consumption growth is high, real interest rates on government bonds tend to be low, even after controlling for a broad array of macroeconomic variables. This suggests that this co-movement is systematically connected to real interest rates. In that sense, the paper also suggests a new channel through which monetary policy may have been a determinant of the secular decline in real interest rates.

[2] For example, as of 2015, the share of public debt held by domestic creditors is 64 percent in the U.S., 69 percent in the United Kingdom, and 78 percent in Canada. Aizenman and Marion (2011) report that the share of U.S. public debt held in Treasury Inflation-Protected Securities (TIPS) is less than 8 percent, as of 2009.

Notice, though, that the same logic that makes nominal debt more attractive to lenders when inflation is procyclical, also suggests that nominal debt is less attractive to borrowers. Consider, for example, a recession which is also accompanied by deflation. In that state, the lenders are happy to receive a large payoff on their nominal assets at the time when their consumption is low (recession). For this reason, they will demand a lower interest rate when lending to the government as discussed earlier. Consider now though the point of view of the borrower (the government), which has to make larger payments at exactly the time when its income is low. Thus, inflation procyclicality would also tend to make debt crises and default by the borrower more likely, which in turn would abruptly push up interest rates in bad times. In fact, we show that the "procyclicality discount" is stronger in good times.

In the second part of the paper, we develop a simple model of debt and default with stochastic inflation that serves two purposes. The first is to show that the empirical connection between debt pricing and inflation dynamics can be understood using simple asset pricing logic. The second is to analyze how the original asset pricing logic for debt pricing and its dynamics is modified in the presence of sovereign default risk.[3]

Our model extends existing models of sovereign debt in two directions. First, we introduce domestic risk averse lenders, in contrast to the common assumption of foreign risk neutral lenders in the literature on sovereign debt crises in emerging economies. This distinction is important since a large amount of public debt is held domestically in advanced economies.[4] Second, we introduce exogenous stochastic inflation so that government bond rates endogenously reflect both inflation risk and default risk. These two features allow us to explicitly analyze the endogenous connection between stochastic discount factors of the domestic lenders, debt pricing, and default probabilities, and to analyze how this relation changes as the co-movement between inflation and consumption growth varies.

Consistent with the data, our model predicts that borrowing costs fall as the covariance of inflation and consumption growth increases. During normal times, relative to its countercyclical counterpart, the procyclical inflation economy sustains similar levels of debt, exhibits

[3]Sovereign default risk, reflected in credit default swap (CDS) spreads, was material even among developed economies during the European Debt Crisis. For instance, Belgium, Greece, Italy, Portugal, and Spain had CDS-implied default probabilities that exceeded 5 percent in 2011 and 2012.

[4]One could also consider the more intricate case where foreign risk averse lenders experience inflation that is correlated with domestic inflation through financial and trade linkages.

higher default risk, and yet enjoys lower borrowing costs. The procyclicality discount reflects the overall reduction in risk perceived by domestic lenders.

However, for a domestic government, debt becomes less attractive in bad times: deflation makes real government obligations larger in recessions, when the government values consumption more. In contrast, the government in the countercyclical economy benefits from the haircut induced by inflation in bad times: a form of partial default. Thus, the cyclicality of inflation affects the government's incentives to borrow or to repay. In our calibrated model, default risk — and thereby the sovereign real rate — spikes markedly and by more during bad times in the the procyclical economy, compared to the countercyclical economy.

Our paper also has implications for the debate on the costs and benefits of joining or exiting a monetary union. Suppose that the union goes into a recession where some, but not all, members of the union get into fiscal trouble. Then the countries in fiscal trouble would like a more countercyclical monetary policy while the others don't: the contrast over monetary policy increases in a recession. Our paper also suggests that monetary policy may have contributed to the secular decline in real interest rates, while also increasing the likelihood of debt crises during bad times.

Related literature. Our paper is related to several strands of literature. On the empirical side, our findings are related to studies on the importance of the inflation risk premium and its variation, as, for example, Boudoukh (1993) or Ang et al. (2008). Campbell et al. (2009), and Du et al. (2016) focus on the covariance of bond returns and other macroeconomic variables such as consumption growth and equity returns. Song (2016) sheds light on the nature of inflation risk in U.S. bonds markets by estimating a model with time variations in the stance of monetary policy as well as the co-movement of inflation and economic activity. On the theoretical side, the backbone of our set-up is a debt default model with incomplete markets as in Eaton and Gersovitz (1981), Aguiar and Gopinath (2006), or Arellano (2008). While these papers focus on foreign debt, Reinhart and Rogoff (2011) suggest that the connection between default, domestic debt, and inflation is an important one. D'Erasmo and Mendoza (2012), Pouzo and Presno (2014), and Arellano and Kocherlakota (2014) tackle the issue of default on domestic debt but do not include inflation.[5] Araujo et al. (2013), Sunder-Plassmann (2016), Mallucci (2015) and Fried (2017) study how the currency composition of debt interacts with default crises in emerging economies while Berriel and Bhattarai (2013), Faraglia et al. (2013), and Ottonello and Perez (2016) study nominal debt with inflation, in the absence of default. Kursat Onder and Sunel (2016) and Nuño and Thomas (2016) consider the interaction of inflation and default on foreign investors.[6] Our paper is also related to Lizarazo (2013) who studies default in the context of risk averse international lenders. Aguiar et al. (2016) provide an excellent compendium on modeling risk averse competitive lenders in the sovereign default literature.

Our general question is also related to recent work that studies how joining a monetary union can affect the probability of a self-fulfilling crisis in a debt default model (see Aguiar et al. 2013 and Corsetti and Dedola 2016). We complement these papers by highlighting how the cyclicality of inflation impacts fundamental-driven default crises, suggesting a promising extension of existing models of self-fulfilling debt crises.

The paper is structured as follows. In section 2, we discuss our empirical findings. In section 3, we develop a simple model of domestic nominal debt, where we discuss the main mechanisms at work. In section 4, we build a quantitative model of domestic debt and default, and section 5 presents our main results on the impact of inflation cyclicality. Section 6 concludes.

2 Inflation and Real Interest Rates

In this section, we study the empirical relation between several conditional moments of inflation and real interest rates on government debt. The main novel finding of the section is that higher covariance of inflation with economic activity is robustly and significantly

[5] Broner et al. (2010) examine the role of secondary asset markets which make the distinction between foreign and domestic default less stark.

[6] Much of the existing literature has focused on strategic inflation, even hyperinflation, as a countercyclical policy option that governments with limited commitment can use when faced with a high debt burden in bad times. That focus is certainly legitimate for emerging economies but less warranted in the context of advanced economies mainly because of monetary policy independence. Central bank independence makes the cost of hyperinflation higher than the cost of a default, perhaps due to limited participation in bond markets in contrast to the economy-wide effects of hyperinflation.

associated with lower real interest rates on government debt.

Our dataset includes quarterly observations on real consumption growth, inflation, interest rates on government bonds, and government debt-to-GDP ratios for an unbalanced panel of 19 OECD economies from 1985Q1 to 2015Q4. The countries in the dataset are: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Italy, Japan, Korea, Netherlands, Norway, Portugal, Spain, Sweden, Switzerland, United Kingdom, and the United States.

We mainly use quarterly data from the IMF and the OECD to document our empirical findings. We compute inflation as the change in log GDP deflator using data from the OECD. We use nominal interest rates on government bonds from the IMF International Financial Statistics (IFS). For government debt, we use quarterly series from Oxford Economics on gross government debt relative to GDP, extended with quarterly OECD data on central government debt relative to GDP. Quarterly real consumption is constructed as the sum of private and public real consumption using the data from the OECD.

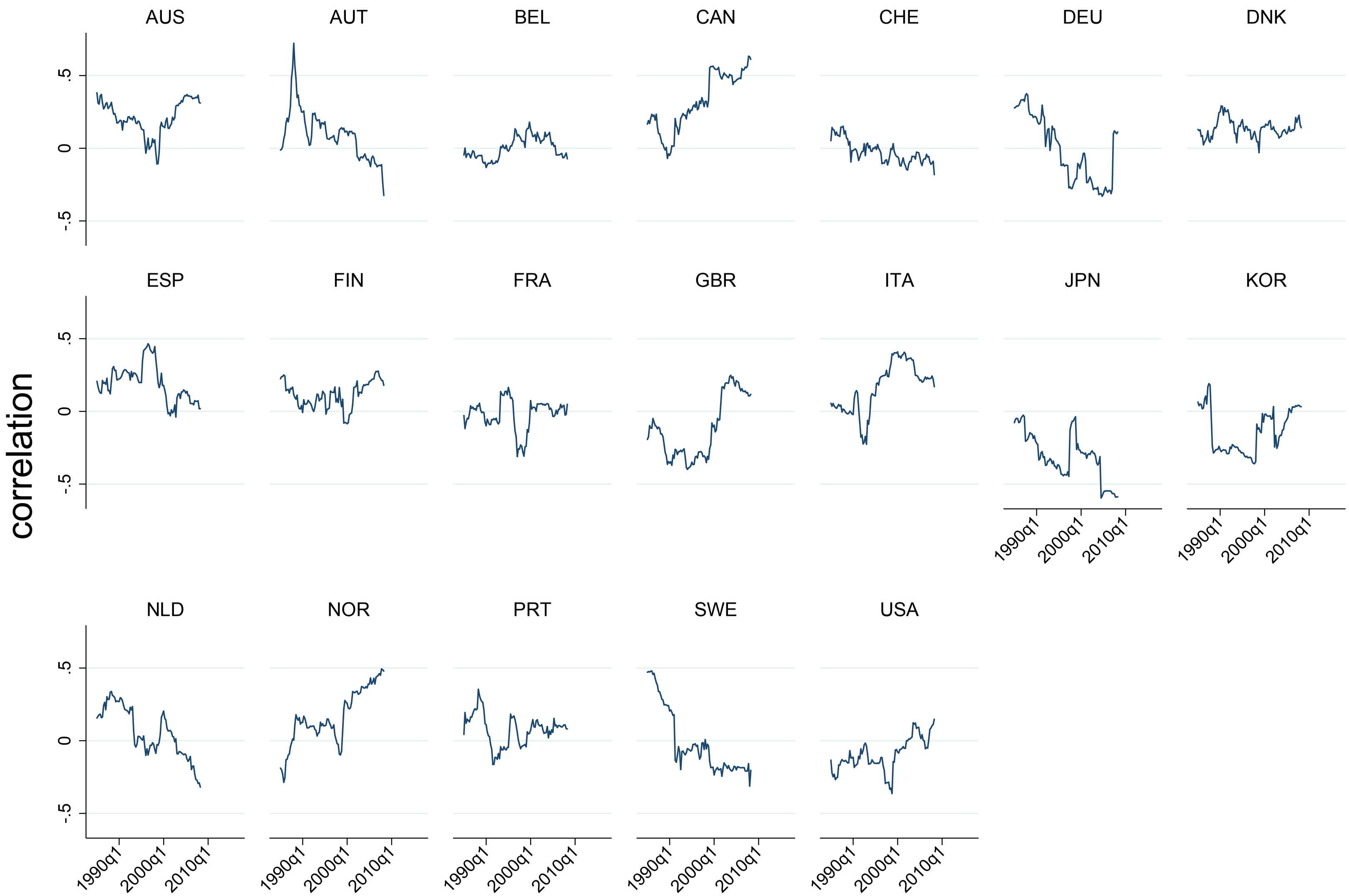
Using this data, we construct real interest rates using expected inflation and the conditional co-movement between inflation and consumption growth. To do, so we follow Boudoukh (1993) and first formulate a vector auto-regression (VAR) model for inflation and consumption growth. The basic VAR is:

$$\begin{bmatrix} \pi_{it} \\ g_{it} \end{bmatrix} = \mathbf{A_i} \begin{bmatrix} \pi_{it-1} \\ g_{it-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{\pi it} \\ \varepsilon_{git} \end{bmatrix} \tag{1}$$

where π_{it} is inflation, and g_{it} is the change in log consumption in country i in period t, A_i is a country-specific 2-by-2 matrix, and $\varepsilon_{\pi it}$ and ε_{git} are innovations in the two time series. We then estimate the VAR using standard OLS and construct time series for residuals $\varepsilon_{\pi it}$ and ε_{git} for each country.

We measure the expected inflation as the forward-looking predicted inflation from the VAR, that is $\mathbf{E}[\pi_{i,t+1}]$. We then derive real rates on government debt as nominal rates less expected inflation. Finally, we measure the conditional co-movement between inflation and consumption growth by measuring the co-variance between the two innovations, $\varepsilon_{\pi it}$ and ε_{git}, in overlapping country-windows, comprised of 40 quarters.

Figure 2: Conditional correlation between inflation and consumption growth



In Figure 2, we plot the path of the conditional correlation for the countries in our sample. It illustrates that the co-movement of inflation and consumption growth varies over time and across countries. In many countries, such as Canada, Italy, Norway, the U.S., or the U.K., the co-movement of inflation and consumption growth has clearly increased since the mid-1980s, while it has sharply decreased or fluctuated in other countries such as Germany.

With this dataset, we estimate how the conditional covariance of inflation and consumption growth relates to interest rates faced by governments. All specifications include a full set of country and time fixed effects.

In Table 1, we regress the real interest rate on the conditional covariance of inflation with consumption growth. The main result from Table 1 is that in periods with higher conditional covariance between inflation and consumption growth, governments face lower interest rates. This finding is robust to the inclusion of the level of government debt and average inflation and consumption growth (column 2). This association is also robust to the inclusion of the variances of inflation and consumption growth as additional regressors (columns 3). In the appendix, we also show that the results are robust to using different yield measures and different debt measures.

Overall, our results show that the co-movement of inflation and consumption growth are associated with lower real interest rates that governments face. We call this the *inflation procyclicality discount*. The magnitude of this discount is economically significant. As an illustration of its magnitude, consider moving from a country/time period in which the inflation/consumption correlation is around -0.3 (for example, the U.S. in the 1980s) to a sample period in which the correlation is around 0.1 (for example, the U.S. in the 2000s). This roughly corresponds to a change in correlation equal to two times the standard deviation of correlation in our sample. Using the coefficient estimated in column (4) of Table 1, we can see that this increase in procyclicality is associated to a lowering of real rates by 42 basis points. Similarly, using the coefficient estimated in column (3) of Table 1, a fall in covariance that is twice as large as our sample standard deviation, is associated with a decrease in real rates by 61 basis points.

Our second main finding is that the procyclicality discount is a good times discount. This can be seen in Table 2, which includes an indicator for good times, defined to be a (10-year)

Table 1: Inflation consumption growth co-movement and real interest rates

	Real yield on government debt			
	covariance			correlation
	(1)	(2)	(3)	(4)
Inflation consumption co-movement	-1.797***	-1.637***	-1.804**	-1.062**
	(0.539)	(0.380)	(0.636)	(0.432)
Lagged government debt	0.017***	0.015***	0.015***	0.015***
	(0.005)	(0.005)	(0.005)	(0.005)
Average inflation residual l		2.413**	2.139*	1.908*
		(0.985)	(1.021)	(0.927)
Average cons. growth residual		-1.748	-1.649	-1.516
		(1.072)	(1.038)	(1.079)
Variance of inflation residual			0.296	0.257
			(0.291)	(0.311)
Var. of cons. growth residual			-0.0574	0.233
			(0.184)	(0.118)
standard deviation of co-movement	0.170	0.170	0.170	0.210
adj. R^2	0.881	0.902	0.903	0.903
N	1726	1726	1726	1726

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses. Standard errors clustered by country. All regressions include country and time fixed effects. The data is a quarterly unbalanced panel from 1985Q1 to 2015Q4 including AUS, AUT, BEL, CAN, CHE, DEU, DNK, ESP, FIN, FRA, GBR, ITA, JAP, KOR,NLD, NOR, POR, SWE, USA. All variables are computed over a forward-looking ten-year window. The co-movement of inflation and consumption growth is measured as the covariance of residuals within that window: $\mathbf{cov}_t(\varepsilon_{\pi it}, \varepsilon_{git})$. Other regressors are averages and variances of those residuals in the window and lagged debt.

window in which the average residual consumption growth is positive, and its interaction with the covariance of inflation and consumption growth. Column (2) shows that the interaction term is negative and statistically significant, while the unconditional procyclicality discount is no longer statistically significant, implying that the inflation cyclicality discount is a good times discount. Finally, column (3) shows that when we also include the interaction of covariance and an indicator for bad times (the complement of good times), the good times interaction term becomes more negative and statistically significant, while the bad times interaction term is not statistically significant.

Overall, we find that procyclical inflation episodes are associated with a significant dis-

Table 2: Inflation procyclicality discount in good times

	Real yield on government debt		
	(1)	(2)	(3)
Inflation consumption covariance	-1.804**	-1.159	
	(0.636)	(0.683)	
Indicator(good times)		-0.230	-0.230
		(0.227)	(0.227)
Interaction term (good times)		-1.834***	-2.994***
		(0.506)	(0.696)
Interaction term (bad times)			-1.159
			(0.683)
Other controls	Yes	Yes	Yes
adj. R^2	0.903	0.910	0.910
N	1726	1726	1726

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses. Standard errors clustered by country. All regressions include country and time fixed effects. The data is a quarterly unbalanced panel from 1985Q1 to 2015Q4 including AUS, AUT, BEL, CAN, CHE, DEU, DNK, ESP, FIN, FRA, GBR, ITA, JAP, KOR, NLD, NOR, POR, SWE, USA. All variables are computed over a forward-looking ten-year window. The co-movement of inflation and consumption growth is measured as the covariance of residuals within that window: $\mathbf{cov}_t(\varepsilon_{\pi it}, \varepsilon_{git})$. Other regressors are averages and variances of those residuals in the window and lagged debt.

count on real sovereign yields albeit such "inflation procyclicality discount" vanishes in bad times. The standard consumption-based asset pricing model suggests that the hedging benefits of procyclical inflation rationalize an inflation procyclicality discount. However, the state-dependent nature of the procyclicality discount suggests that bad times are associated with additional credit risk, possibly default risk. From the government's perspective, inflation procyclicality is not desirable in bad times *ceteris paribus* and reduces the government's willingness to pay. In the next section, we develop a simple theory to understand the relation between the inflation-consumption growth co-movement, interest rates, and default.

3 Simple Model

In this section we highlight the main economic mechanism of this paper through a very simple two-period model of inflation and default, where equilibrium outcomes can be characterized using simple diagrams.

3.1 Simple model without default

Consider a two-period, one-good, closed economy with competitive lenders and borrowers. Both borrowers and lenders receive one unit of the good in the first period and an endowment of x in the second period, where x is a random variable with c.d.f. F over X, with finite support $X = [x_{\min}, x_{\max}]$ and $\mathbf{E}(x) = \mu$. The variable x here captures aggregate risk of the economy, to which both lenders and borrowers are exposed. We assume that the only difference between lenders and borrowers (i.e. motive to intertemporal trade) lies in their preferences. In particular, we assume that $\beta_\ell > \beta_b$ are the discount factors of lenders and borrowers, respectively. Lenders and borrowers can trade a nominal bond at price q today, which pays a nominal amount of 1 tomorrow. We normalize the current price level to 1, and assume that the future price level is given by $1 + \pi(x; \kappa) \equiv [1 + \kappa(\mu - x)]^{-1}$, where κ is the key parameter, capturing the cyclicality of inflation. If $\kappa > 0$, prices (and inflation) are procyclical, so the bond pays less in good states (when x is high) of the world, while the reverse is true if $\kappa < 0$.

The borrower solves

$$\max_{b_b} u(1 + qb_b) + \beta_b \int_X u\left(x - \frac{b_b}{1 + \pi(x; \kappa)}\right) dF(x), \tag{2}$$

and the lender solves

$$\max_{b_\ell} u(1 - qb_\ell) + \beta_\ell \int_X u\left(x + \frac{b_\ell}{1 + \pi(x; \kappa)}\right) dF(x), \tag{3}$$

Notice that both borrowers and lenders act competitively so that they take the price of bonds as given. An equilibrium is then simply a bond price and bond quantities of borrowers and lenders such that given prices the bond quantities are optimal for each agent and the bond market clears.

Theorem 1 shows that, under certain conditions, we can demonstrate an inflation cyclicality discount arising from the hedging benefits of inflation procyclicality.

Theorem 1. Inflation procyclicality discount

Assume that both borowers and lenders have quadratic utility, i.e. $u(c) = Ac - \frac{\phi}{2}c^2$ *with* $\frac{A}{\phi} > \max\{1, \mu\}$. *For* $-\underline{\kappa} \leq \kappa \leq \overline{\kappa}$, *with* $\underline{\kappa} > 0$ *and* $\overline{\kappa} > 0$ *specified in Appendix B.1, there is an inflation procyclicality discount. That is,*

$$\frac{\partial q}{\partial \kappa} > 0. \tag{4}$$

Proof: See Appendix B.1.

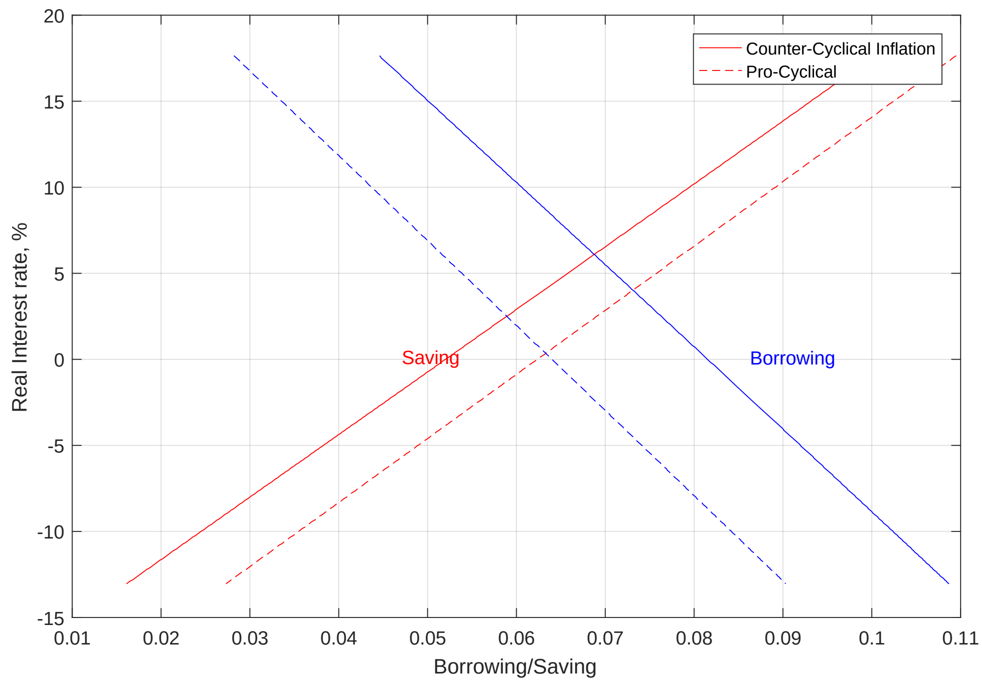
The proof involves showing that the lender's supply of savings increases with κ while the borrower's demand for borrowing decreases with κ. Figure 3 provides some visual intuition for this result. The lines in the figure represent the desired supply of loans from the lender (increasing in the interest rate) and the desired demand loans by the borrower (decreasing in the interest rate). The solid lines are supply and demand with counter-cyclical inflation, while the dashed lines are supply and demand with pro-cyclical inflation. Note that as inflation goes from counter to pro-cyclical the supply of loans increases, highlighting the hedging effect. While procyclical inflation makes debt less risky for the lender, the opposite is true for the borrower, and this results in the decrease in the demand for borrowing. Since supply increases and demand falls, the equilibrium interest rate unequivocally falls, while the equilibrium level of debt can move in either direction.

3.2 Simple model with default

Now consider the possibility that the nominal contract can be defaulted on. IN particular the borrower can default on its bond payments and if it does so, no payments are made and it incurs a cost $C(x) = \psi(x - x_{\min})^2$. As in (Dubey et al. 2005) we keep the assumption of competitive borrowers, so they do not perceive that their borrowing decision affect the equilibrium interest rate they face. In this environment, there will be equilibrium default when default costs are below repayment, hence the default set $\widehat{X}$ is given by

$$x : C(x) < \frac{b_b}{1 + \pi(x; \kappa)} \tag{5}$$

Figure 3: Interest rates and cyclicality of inflation



which typically is an interval, i.e. default happens when income is low enough and when debt is high enough. The key observation is that in a world with default, the cyclicality of inflation can change the default set, thereby altering the hedging properties of bonds. Theorem 2 shows that, under certain regularity conditions, the default set $\widehat{X}$ increases with the level of debt (b_b) and the cyclicality of inflation (κ).

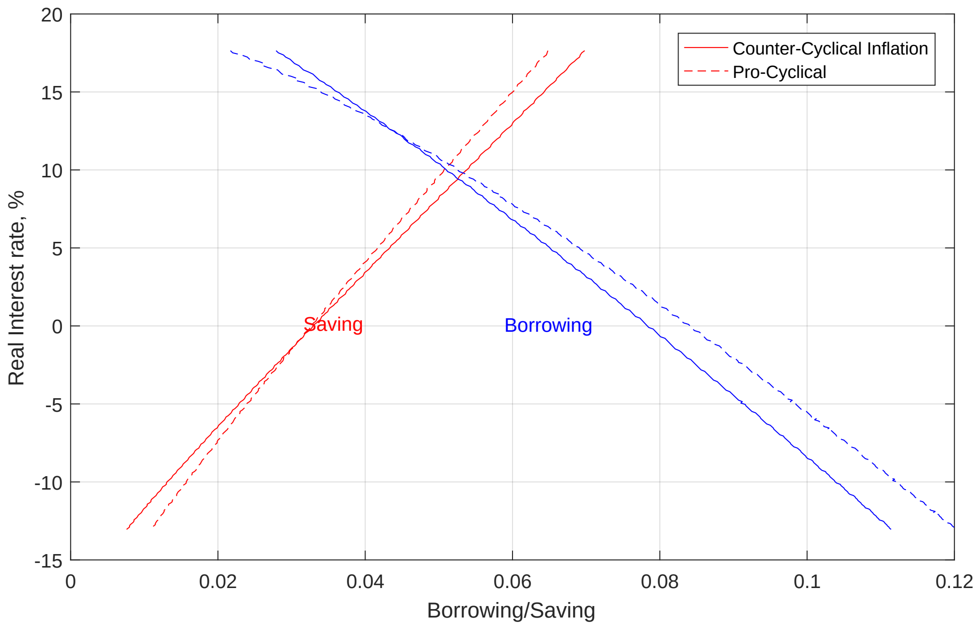
Theorem 2. Inflation procyclicality and default

Assume that $-(\mu - x_{\min})^{-1} < \kappa < (x_{\max} - \mu)^{-1}$ *and* $\psi(\mu - x_{\min})^2 > \overline{B}(\kappa)$ *where* $\overline{B}(\kappa)$ *is defined in Appendix B.2. Then there exists a unique threshold* $\widehat{x}(\kappa, b_b) \in [x_{\min}, \mu]$ *such that default occurs if and only if* $x \in [x_{\min}, \widehat{x}]$*. Furthermore, the default threshold is increasing in debt* (b_b) *and the cyclicality of inflation* (κ)*, ceteris paribus. That is,*

$$\frac{\partial \widehat{x}(\kappa, b_b)}{\partial b_b} > 0 \tag{6}$$

$$\frac{\partial \widehat{x}(\kappa, b_b)}{\partial \kappa} > 0. \tag{7}$$

Figure 4: Interest rates and cyclicality of inflation



Proof: See Appendix B.2.

Given this result we can then write the problem of the borrower as

$$\max_{b_b} u\left(1+qb_b\right)+\beta_b\left(\underbrace{\int_{\widehat{x}(b,\kappa)}^{x_{\max}} u\left(x-\frac{b_b}{1+\pi(x)}\right)}_{\text{Repayment}}+\underbrace{\int_{x_{\min}}^{\widehat{x}(b,\kappa)} u\left(x-C(x)\right)}_{\text{Default and suffer cost}}\right) dF\left(x\right) \tag{8}$$

The lender, taking as given the default threshold $\widehat{x}$, solves

$$\max_{b_\ell} u\left(1-qb_\ell\right)+\beta_\ell\left(\underbrace{\int_{\widehat{x}}^{x_{\max}} u\left(x+\frac{b_\ell}{1+\pi(x)}\right)}_{\text{Repayment}}+\underbrace{\int_{x_{\min}}^{\widehat{x}} u\left(x\right)}_{\text{Defaulted on}}\right) dF\left(x\right). \tag{9}$$

In the model with default, changes in covariance lead to changes not only to quantities but also to the default threshold, complicating the analysis. Thus, to gain further intuition, we utilize a numerical illustration. Figure 4 shows that, unlike the model without default in which higher inflation procyclicality unequivocally reduced interest rates, in the model with default, higher inflation procylicality can increase real rates. This is because countercyclical inflation, which implies low repayments in bad states, is substitute with default, while pro-cyclical inflation, which implies high repayments in bad states, is complement with default. Thus, a country following pro-cyclical inflation will face a lower interest rate if not at default risk, but might face a sudden spike in rates in bad times. This simple model motivates our quantitative model, in which the cyclicality of inflation is a key driver of real sovereign yields, nominal debt dynamics, and default risk.

4 Quantitative Model

We extend the standard sovereign default model of Eaton and Gersovitz (1981) and Arellano (2008) in two dimensions: exogenous *inflation* and risk averse *domestic lenders*.

4.1 Environment

We consider a closed economy that is populated by a continuum of households who lend to the government. Both the representative household and the government have preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta_i^t u_i(c_{it}) \tag{10}$$

where $0 < \beta_g < \beta_\ell < 1$ and $c_{gt}, c_{\ell t}$ are the discount factors and consumption at time t of the government and the household respectively. Both agents' period utility function is given by

$$u_i(c) = \frac{c^{1-\gamma_i}}{1-\gamma_i} \tag{11}$$

where γ_g and γ_ℓ represent the risk aversion of the government and households respectively. The households receive a stochastic stream of non-storable consumption good y, which is taxed by the government at rate $0 < \tau < 1$.

4.2 Government

The government has access to debt markets in which it issues one-period non-contingent bonds to the domestic lenders. Bonds are risky because debt contracts are not enforceable, which may lead to government default, and also because they may lose value due to exogenous inflation. Stochastic endowments y and inflation π follow a joint Markov Process. Let us denote $s = (y, \pi)$. Given the option to default, $V^o(B, s)$ satisfies

$$V^o(B, s) = \max_{c,d} \left\{ V^c(B, s), V^d(B, s) \right\} \tag{12}$$

where B is incoming government assets, V^c is the value of not defaulting, and V^d is the value of default.

When the government defaults, the economy is in financial autarky for a stochastic number of periods and income may fall. Upon reentry, after k periods, the government's debt obligation is $-\lambda^k B$ where $1 - \lambda$ is the rate at which the government's debt obligation decays each period. This tractable way of modeling partial default is also consistent with the fact that longer default episodes are associated with lower recovery rates, as documented by Benjamin and Wright (2009). Setting $\lambda = 0$ corresponds to the model with full default.

The government's value of default is then given by

$$V^d(B, s) = u_g \left(\tau(y - \phi^d(y)) \right) + \beta_g \mathbf{E}_{s'|s} \left[\theta V^o \left(\frac{\lambda B}{1 + \pi'}, s' \right) + (1 - \theta) V^d \left(\frac{\lambda B}{1 + \pi'}, s' \right) \right] \tag{13}$$

where $0 < \theta < 1$ is the probability that the government will regain access to credit markets, and $\phi^d(y)$ is the loss in income during default. In particular, we assume a quadratic function

$$\phi^d(y) = \max \left\{ 0, \frac{d_1}{d_0} y + \left(d_1 - \frac{d_1}{d_0} \right) y^2 \right\}, \tag{14}$$

similar to Chatterjee and Eyigungor (2013), except that the expression has been written such that d_1 is the default cost at mean output ($y = 1$) and d_0 is the deviation from mean output above which the default costs are positive (that is, $y > 1 + d_0$).

The value of not defaulting, is given by

$$V^c(B,s) = \max_{B' \le 0} \left\{ u_g \left(\tau y - q(B,s,B')B' + B \right) + \beta_g \mathbf{E}_{s'|s} \left[V^o \left(\frac{B'}{1+\pi'}, s' \right) \right] \right\} \tag{15}$$

where $q(B,s,B')$ is the bond price schedule the government faces. Note that the real return on government debt is stochastic, even in the absence of default, due to inflation risk.

4.3 Households

Households take as given the price of bonds, $q^*(B,s)$, and the policy functions for government assets, $B^*(B,s)$, and default, $d^*(B,s)$. If the government does not have access to credit markets, the lender's value function is given by

$$\begin{aligned} W^d(b;\ B,s) = \max_{b'}\ & u_\ell \left((1-\tau)(y - \phi^d(y)) - q^d(B,s)\,(b'-b) \right) \\ & +\ \beta_\ell \mathbf{E}_{s'\,|\,s} \left\{ \begin{array}{c} \theta \left(1 - d^* \left(\frac{\lambda B}{1+\pi'}, s' \right) \right) W^c \left(\frac{\lambda b'}{1+\pi'};\ \frac{\lambda B}{1+\pi'}, s' \right) \\ + \left(1 - \theta + \theta d^* \left(\frac{\lambda B}{1+\pi'}, s' \right) \right) W^d \left(\frac{\lambda b'}{1+\pi'};\ \frac{\lambda B}{1+\pi'}, s' \right) \end{array} \right\} \end{aligned} \tag{16}$$

where q^d is the price of a bond in default and W^c is the household's value function when the government has access to credit markets, which is given by

$$\begin{aligned} W^c(b;\ B,s) = \max_{b'}\ & u_\ell \left((1-\tau)y + b - q^*(B,s)b' \right) \\ & +\ \beta_\ell \mathbf{E}_{s'|s} \left\{ \begin{array}{c} \left(1 - d^* \left(\frac{B^*(B,s)}{1+\pi'}, s' \right) \right) W^c \left(\frac{b'}{1+\pi'};\ \frac{B^*(B,s)}{1+\pi'}, s' \right) \\ + d^* \left(\frac{B^*(B,s)}{1+\pi'}, s' \right) W^d \left(\frac{b'}{1+\pi'};\ \frac{B^*(B,s)}{1+\pi'}, s' \right) \end{array} \right\} \end{aligned} \tag{17}$$

where $q^*(B,s) = q(B,s,B^*(B,s))$. Note that the first term inside the expectation operator in (16) represents the continuation value in the case that government regains access to credit next period and does not immediately default. The second term represents the continuation value in the case that either the government does not regain access to credit markets or immediately defaults upon regaining access to credit. In both cases, the lender's real assets decay by $1-\lambda$.

4.4 A political economy interpretation

The current model setup, in which the government maximizes the expected utility derived from its own consumption, is equivalent to a political economy model, in which the government borrows from a mass μ of domestic risk averse households who are rich and patient, and maximizes the welfare of a median voter group composed by a unit mass of hand-to-mouth risk averse households who are α-times poorer and impatient. When $\tau = (1 + \alpha\mu)^{-1}$, the two models are equivalent.

4.5 Recursive equilibrium

Definition. The *recursive equilibrium* for this economy is defined as a set of (i) policy functions for household assets $b^*(b; B, s)$, (ii) policy functions for government assets $B^*(B, s)$ and default $d^*(B, s)$, and (iii) price functions $q(B, s, B')$ and $q^d(B, s)$ such that:

1. Taking as given government policies and bond prices, the representative household's policy functions solve the optimization problem in (16) and (17).

2. Taking as given the bond price schedule, the government's policy functions solve the optimization problem in (12), (13), and (15).

3. The bond market clears,
$$b^*(B; B, s) + B^*(B, s) = 0. \tag{18}$$

4. No bond trading in default,
$$b^*(b; B, s) = b. \tag{19}$$

4.6 Characterization

In this environment, the bond price schedule satisfies

$$\begin{aligned} q(B, s, B') \quad = \quad & \beta_\ell \mathbf{E}_{s'|s} \left[\frac{1 - d^*\left(\frac{B'}{1+\pi'}, s'\right)}{1+\pi'} \frac{u'_\ell\left(c^*_\ell(\frac{B'}{1+\pi'}, s')\right)}{u'_\ell\left(c_\ell(B, s, B')\right)} \right] \\ & + \beta_\ell \mathbf{E}_{s'|s} \left[\frac{d^*\left(\frac{B'}{1+\pi'}, s'\right)}{1+\pi'} q^d \left(\frac{B'}{1+\pi'}, s' \right) \frac{u'_\ell\left((1-\tau)(y'^d(y'))\right)}{u'_\ell\left(c_\ell(B, s, B')\right)} \right] \end{aligned} \tag{20}$$

where the state-contingent consumption of the lender is

$$c_\ell(B,s,B') = (1-\tau)y - B + q(B,s,B')B' \tag{21}$$

$$c_\ell^*(B,s) = (1-\tau)y - B + q^*(B,s)B^*(B,s) \tag{22}$$

The bond price schedule $q(B,s,B')$ reflects a penalty (discount) for future inflation (deflation) $(1+\pi)$, a default premium for future default $d^*\left(\frac{B'}{1+\pi'},s'\right)$, and the lender's stochastic discount factor (SDF), which is also endogenous since it varies with the debt policy function $B^*(\cdot)$.

Similarly, the default bond price is

$$\begin{aligned} q^d(B,s) &= \beta_\ell \lambda \theta \mathbf{E}_{s'|s}\left[\frac{1-d^*\left(\frac{\lambda B}{1+\pi'},s'\right)}{1+\pi'}\frac{u_\ell'\left(c_\ell^*(\frac{\lambda B}{1+\pi'},s')\right)}{u_\ell'\left((1-\tau)(y-\phi^d(y))\right)}\right] \\ &\quad +\beta_\ell \lambda \mathbf{E}_{s'|s}\left[\left(1-\theta+\theta d^*\left(\frac{\lambda B}{1+\pi'},s'\right)\right)\frac{q^d\left(\frac{\lambda B}{1+\pi'},s'\right)}{1+\pi'}\frac{u_\ell'\left((1-\tau)(y'^d(y'))\right)}{u_\ell'\left((1-\tau)(y-\phi^d(y))\right)}\right]. \end{aligned} \tag{23}$$

It is convenient to further decompose the bond pricing formula in order to highlight the different channels in the bond price. By using the definition of covariance, we can expand the expectation of the product of future deflation, repayment probabilities, and the lender's SDF to obtain

$$\begin{aligned} q(B,s,B') &= \beta_\ell \mathbf{E}_{s'|s}\left[\frac{1}{1+\pi'}\right]\mathbf{E}_{s'|s}\left[1-d^*\left(\frac{B'}{1+\pi'},s'\right)\right]\mathbf{E}_{s'|s}\left[\frac{u_\ell'\left(c_\ell^*(\frac{B'}{1+\pi'},s')\right)}{u'\left(c_\ell(B,s,B')\right)}\right] \\ &\quad +\beta_\ell \mathbf{E}_{s'|s}\left[1-d^*\left(\frac{B'}{1+\pi'},s'\right)\right]\mathbf{cov}_{s'|s}\left[\frac{1}{1+\pi'},\frac{u_\ell'\left(c_\ell^*(\frac{B'}{1+\pi'},s')\right)}{u_\ell'(c_\ell(B,s,B'))}\right] \\ &\quad +\beta_\ell \mathbf{E}_{s'|s}\left[\frac{1}{1+\pi'}\right]\mathbf{cov}_{s'|s}\left[1-d^*\left(\frac{B'}{1+\pi'},s'\right),\frac{u_\ell'\left(c_\ell^*(\frac{B'}{1+\pi'},s')\right)}{u_\ell'(c_\ell(B,s,B'))}\right] \\ &\quad +\beta_\ell \mathbf{E}_{s'|s}\left[\frac{u_\ell'\left(c_\ell^*(\frac{B'}{1+\pi'},s')\right)}{u_\ell'(c_\ell(B,s,B'))}\right]\mathbf{cov}_{s'|s}\left[\frac{1}{1+\pi'},1-d^*\left(\frac{B'}{1+\pi'},s'\right)\right] \\ &\quad +\beta_\ell \mathbf{E}_{s'|s}\left[\frac{d^*\left(\frac{B'}{1+\pi'},s'\right)}{1+\pi'}q^d\left(\frac{B'}{1+\pi'},s'\right)\frac{u_\ell'\left((1-\tau)(y'^d(y'))\right)}{u_\ell'\left(c_\ell(B,s,B')\right))}\right]. \end{aligned} \tag{24}$$

The first line in (24) shows that the conditional probability of default and expected inflation increase borrowing costs — effects which are standard but are now endogenous to the cyclicality of inflation.

The second line shows the standard direct effect of inflation cyclicality: inflation procyclicality *reduces* borrowing costs through the hedging benefit of the co-movement of inflation and consumption growth.

The third line shows that the countercyclicality of default also increases borrowing costs since the lenders are domestic and value consumption more in the bad states that coincide with default. The fourth line shows how the co-movement of default and inflation affects borrowing costs: in the procyclical economy, defaults tend to coincide with surprise deflation, eliminating what would have been high returns for the lender.

Overall, equation (24) elicits the intuition from equation (**??**) in the simple model: the cyclicality of inflation in a model with domestic default entails various endogenous channels including, but not limited to, an endogenous default risk and the standard hedging argument. The interplay between these channels also vary over the cycle: inflation procyclicality is likely to be associated with a discount when default risk is low, but not in bad times as default motives are increased with inflation procyclicality. We turn to a quantitative analysis of these forces in the next section and use the model to assess the implications of the ''inflation procyclicality discount'' we documented.

5 Quantitative analysis

In this section, we use a calibrated version of the model to investigate the role of the inflation process on the dynamics of interest rates, debt, and default crises. In particular, we calibrate the model with zero covariance to match conditional default frequencies and the spread in advanced economies. We then use the covariance estimates from section 2 to assess the impact of different inflation processes on interest rates, debt dynamics, and default crises.

5.1 Functional forms and calibration

Endowments y and inflation π follow a joint process:

$$\begin{bmatrix} \log y' \\ \pi' \end{bmatrix} = \begin{bmatrix} \rho_{y,y} & \rho_{\pi,y} \\ \rho_{y,\pi} & \rho_{\pi,\pi} \end{bmatrix} \begin{bmatrix} \log y \\ \pi \end{bmatrix} + \begin{bmatrix} \epsilon_y \\ \epsilon_\pi \end{bmatrix} \tag{25}$$

where

$$\begin{bmatrix} \epsilon_y \\ \epsilon_\pi \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_y^2 & \sigma_{\pi,y} \\ \sigma_{\pi,y} & \sigma_\pi^2 \end{bmatrix} \right).$$

Since we consider a closed economy environment, output in our model is equal to consumption. This allows us to use our VAR results from (1) to guide our parameter values. Specifically, we set the persistence of output $\rho_{y,y}$ to 0.8, the persistence of inflation $\rho_{\pi,\pi}$ to 0.8, the spillover terms $\rho_{y,\pi}$ and $\rho_{\pi,y}$ to zero, and both variance terms σ_y and σ_π to 0.010 based on the parameters estimated for the cross section of OECD economies in our dataset. Table 12 in appendix A contains the detailed estimates by country. We consider two values for the covariance of inflation and output $\sigma_{\pi,y}$: +0.255e-4 and -0.255e-4. These values represent a 1.5 standard deviation increase and decrease from the median of the covariance of inflation and consumption residuals computed at 10 year windows, which is close to zero.

We choose the government's discount factor β_g and income loss parameters d_0 and d_1 to jointly match default probabilities of 1.02 percent in good times and 2.58 percent in bad times, and an average interest rate spread , which is 0.74 percent. The default probabilities are computed using CDS-implied default probabilities for OECD countries between 2001 and 2015 with a bad-times threshold of 1 standard deviation below trend consumption. Sovereign spreads are computed by subtracting the German nominal rate from the nominal rates of the Eurozone economies in our dataset over the same period.

We set the discount factor β_ℓ of the lender to be 0.99 to match an implied risk-free rate of 1 percent. We set the government's risk aversion γ_g to be 2, as is standard in the macro and sovereign debt literature. We set the lender's risk aversion γ_ℓ to be 8, following Storesletten et al. (2007). This higher level of risk aversion of the lender is also common in the finance and equity premium puzzle literature (for example, see Bansal and Yaron 2004 and Mehra and Prescott 1985).

Table 3: Calibration – Baseline economy with acyclical inflation

Parameters	Values	
		(Joint targets)
Gov't discount factor β_h	0.763	Default prob. in good times: 1.02 percent*
Default cost cutoff d_0	-0.037	Average spread: 0.74 percent**
Default cost at mean d_1	0.040	Default prob. in bad times: 2.58 percent*
Persistence $\rho_{y,y} = \rho_{\pi,\pi}$	0.80	VAR estimates (OECD cross section)
Spillovers $\rho_{\pi,y} = \rho_{y,\pi}$	0.00	VAR estimates
Volatility $\sigma_y = \sigma_\pi$	0.01	VAR estimates
Covariance of innovations $\sigma_{\pi,y}$	0.00	acyclical baseline ± 1.5 s.d. $= \pm 0.255$e-4
Lender discount factor β_ℓ	0.99	Risk-free rate: 1 percent
Lender risk aversion γ_ℓ	8	Storesletten et al. (2007)
Gov't risk aversion γ_g	2	
Probability of re-entry θ	0.10	Average exclusion: 10 quarters†
Recovery parameter λ	0.96	Average recovery rate: 50 percent‡
Tax rate τ	0.19	Government consumption (percent GDP)

* : CDS-implied default probabilities 2001-2015. ** : Eurozone nominal rates less German rates 2001-2015. † : See Richmond and Dias (2008). ‡ : See Benjamin and Wright (2009).

The probability of re-entry θ is set to match the average exclusion of 10 quarters as documented by Richmond and Dias (2008) and the recovery parameter λ is set to be consistent with the average recovery rate of 50 percent reported by Benjamin and Wright (2009). To compute the average recovery rate, we take the following steps. First, we consider a default to be over when the government regains access to credit, which on average lasts 10 quarters. Second, we discount the payment back to the period of default by an annualized interest rate of 10 percent as in Benjamin and Wright (2009).

Finally, we set the tax rates to be 19 percent to match the government consumption share of GDP in OECD economies between 1985 and 2015. A summary of our parameters can be found in Table 3.[7]

[7] A tax rate $\tau = .18$ is equivalent to the political economy interpretation of the model described in section 4.4, calibrated to match U.S. income inequality between those with an annual income above $\$40,000$ a year ($\sim$ top 42 percent) and those below ($\sim$ bottom 58 percent). Specifically, this is achieved by setting $\mu = 42.1/57.9 = .73$ as the relative size of the two groups and $\alpha = 110.5/17.5 = 6.31$ as the ratio of their average incomes.

5.2 Results

Using the calibrated model, we contrast two inflation regimes: a procyclical economy and a countercyclical economy, which correspond to a covariance of inflation and consumption innovations of 1.5 standard deviations above and below zero respectively. We report long-run equilibrium outcomes for both regimes in Table 4.

We find that relative to its countercyclical analog, the procyclical economy (i) enjoys lower borrowing costs, (ii) sustains similar levels of debt, and (iii) experiences more default crises.[8]

Table 4: Business cycle statistics

	Positive co-movement (+1.5 s.d.)	Negative co-movement (-1.5 s.d.)
Default rate (percent)	1.31	1.14
Spreads (percent)	0.67	0.71
Debt (percent of borrower income)	5.55	5.57

A further decomposition of the equilibrium bond price based on equation (24) is shown in Table 5. The second row shows the default premium of 17 basis points and the third row shows the direct procyclicality discount of 11 basis points relative to the countercyclical economy. Even though the direct procyclicality discount is small relative to the default premium, the first row shows that equilibrium borrowing costs are still 10 basis points *lower* in the procyclical economy. Such net interest rate discount in the procyclical economy highlights the role of general equilibrium forces in the presence of counterbalancing effects of default and cyclicality hedging.

In the same spirit, one can compute an "adjusted" cyclicality effect defined as the overall discount required to offset the counterfactual default penalty across the two regimes. Taking the difference between the net discount on borrowing costs and the default penalty, we get an adjusted cyclicality effect of [27] basis points. In that sense, the effect of inflation cyclicality on interest rates is amplified through the endogenous effect on default risk.

[8]The calibrated model features low levels of debt for both regimes. This is partly due to our targeted default probabilities, which are somewhat higher than usual since the periods for which we have the CDS-implied default probabilities include the Great Recession and the European Debt crisis.

Table 5: Bond price decomposition: default and cyclicality hedging

	Positive co-movement (+1.5 s.d.)	Negative co-movement (–1.5 s.d.)	Difference (annualized, basis points)
bond price: $100 \times q$	99.09	99.07	+10
no default: $100 \times \mathbf{E}[1-d]$	99.66	99.71	–17
$\mathbf{cov}(\text{irms}, \text{defl.}) \times \mathbf{E}[1-d]$	0.012	-0.016	+11

Even though interest rates are on average lower in the procyclical economy, procyclicality does not always have an overall positive effect of lower rates on interest rates relative to its countercyclical counterpart. When times are good, the procyclical economy enjoys lower borrowing costs despite a higher likelihood of default. When the economy begins to deteriorate, for example with a successive sequence of low output shocks, then the procyclical economy is more likely to face low inflation, or possibly deflation, resulting in an appreciation of the government's real debt obligations, leading to a debt crisis.

In bad times, the procyclical economy faces higher borrowing costs, reflecting the significantly higher probability of default. We classify a period as a bad (good) time in the model if output is below (above) a threshold of 1 standard deviation below mean output. This can be seen in Table 6 where we report average spreads and conditional default probabilities. The table shows that the procyclical economy effectively faces typically lower, but also more volatile, interest rates. In bad times, default risk spikes more sharply in the procyclical economy compared to the countercyclical economy as shown in Table 6. The countercyclical faces smaller surges in default risk partly because inflation innovations shave off some of the debt burden precisely when the default risk is heightened. In that sense, default and inflation are substitutes under countercyclical inflation but complements under procyclical inflation.

5.3 When is procyclicality preferred?

Since the effect of inflation cyclicality on default and interest rates is state-dependent, it is useful to discuss when the government prefers a procyclical regime. In Figure 5, panel (a) shows which cyclicality regime the government prefers across different states of incoming debt

Table 6: Spreads and default risk in good times and bad times

	Positive co-movement (+1.5 s.d.)	Negative co-movement (–1.5 s.d.)	Difference Overall spreads (percent)
Overall spreads (percent)	0.67	0.72	–0.04
Spreads in good times (percent)	0.54	0.60	–0.06
Spreads in bad times (percent)	1.52	1.40	+0.12
Default prob. in good times (percent)	1.05	0.94	+0.11
Default prob. in bad times (percent)	2.81	2.33	+0.48

and inflation-output realizations. Panel (b) in Figure 5 shows the lenders' welfare ranking of the inflation cyclicality regimes.

Figure 5 reveals that the government typically prefers the countercyclical regime, especially in bad states of the world and with high debt. The government only prefers procyclicality in good states of the world with high debt. Despite offering overall lower interest rates, the procyclical inflation regime is not always preferred by the government. This is consistent with the endogenous state- and regime-dependent default premium present in this model. In contrast, the lenders prefer inflation procyclicality almost everywhere, especially in bad states of the world. The lenders' preference for procyclicality align with the government's only in good states of the world.

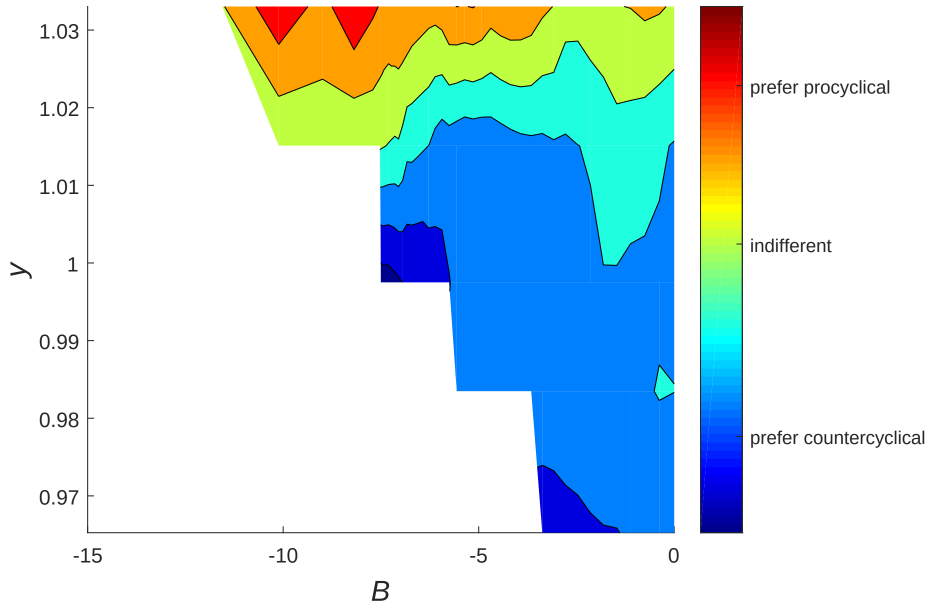
5.4 Robustness with higher lenders' risk aversion

Although the lenders' risk aversion ($\gamma_\ell = 8$) considered in the benchmark calibration is relatively high compared to the standard RBC literature, one can motivate an alternative calibration with an even higher lenders' risk aversion. In fact, one can argue that the large inflation procyclicality discount documented in the data justifies a higher risk aversion, consistent with the macro-finance literature. Such higher risk aversion calibration will not only magnify the standard hedging channel but also make the lenders price default risk differently which in turn affect the government's debt and default choices.

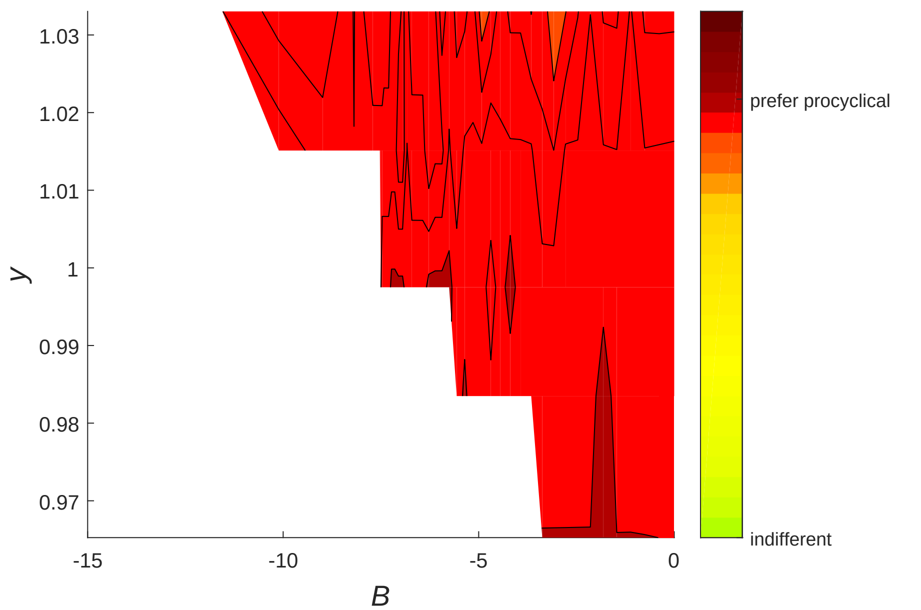
The results from this alternative calibration with a lenders' risk aversion of $\gamma_\ell = 16$ are presented below. Table 7 shows the calibrated parameters under a higher lender risk

Figure 5: Welfare comparison of cyclicality regimes across states

(a) Government



(b) Lender



aversion. Table 8 and Table 9 report the equilibrium outcomes under procyclical inflation and countercyclical inflation.

We find that relative to its countercyclical analog, under higher risk aversion, the pro-

Table 7: Alternative calibration – Baseline economy with acyclical inflation

Parameters	Values	
		(Joint targets)
Gov't discount factor β_h	0.780	Default prob. in good times: 1.02 percent*
Default cost cutoff d_0	-0.025	Average spread: 0.74 percent**
Default cost at mean d_1	0.032	Default prob. in bad times: 2.58 percent*
Lender risk aversion γ_ℓ	16	Benchmark $\times$ 2

All other parameters are the same as in the benchmark calibration presented in Table 3.
* : CDS-implied default probabilities 2001-2015. ** : Eurozone (ex. Greece) nominal rates less German rates 2001-2015. † : See Richmond and Dias (2008). ‡ : See Benjamin and Wright (2009).

cyclical economy (i) enjoys lower borrowing costs, (ii) sustains about a similar debt level, and (iii) experiences fewer default crises. These long-run outcomes are shown in Table 8. Compared to the benchmark calibration, the inflation procyclicality discount is now larger, not just because of a larger hedging effect but also because of a lower default risk on average. Thus, under higher lender risk aversion, endogenous default risk amplifies the procyclicality hedging discount and further lowers interest rates in the procyclical inflation economy.

In bad times, default risk continues to spike more sharply in the procyclical economy however. As shown in Table 9, default risk is higher in the procyclical regime in bad times, resulting in a lower procyclicality discount in bad times. This finding is also consistent with empirical evidence, perhaps making this alternative calibration more appealing. We prefer the the benchmark calibration because it clearly highlights endogenous default risk as it features lower interest rates along with higher default risk in good times.

Table 8: Business cycle statistics – Higher risk aversion

	Positive co-movement (+1.5 s.d.)	Negative co-movement (–1.5 s.d.)
Default rate (percent)	0.91	1.13
Spreads (percent)	0.64	1.10
Debt (percent of borrower income)	2.47	2.42

Table 9: Spreads and default risk across states – Higher risk aversion

	Positive co-movement (+1.5 s.d.)	Negative co-movement (–1.5 s.d.)	Difference Overall spreads (percent)
Overall spreads (percent)	0.64	1.10	–0.46
Spreads in good times (percent)	0.41	0.91	–0.50
Spreads in bad times (percent)	2.05	2.30	–0.25
Default prob. in good times (percent)	0.64	0.92	–0.28
Default prob. in bad times (percent)	2.51	2.43	+0.07

6 Conclusion

The goal of this paper was to investigate how inflation cyclicality affects borrowing costs, and debt and default dynamics. Empirically, we documented that the co-movement of inflation innovations and consumption growth innovations fluctuates over time across a large number of advanced countries. Moreover, we find that increased co-movement of inflation and consumption growth is associated with lower borrowing costs, especially in good times. Theoretically, we showed that the inflation processes—especially inflation cyclicality—can be important in explaining interest rates and the dynamics of default. In particular, in our benchmark calibration, the procyclical inflation economy faces lower borrowing costs, even as default is more likely. However, when the economy deteriorates, the procyclical economy faces a much higher likelihood of facing a debt crisis, because it is more likely to face lower inflation, possibly even deflation, and thus an increasing real debt burden. Our findings have implications for the debate on the costs and benefits of joining or exiting monetary unions. Our findings also suggests a new channel, the interaction of monetary policy and interest rates in the presence of sovereign credit risk, that can help understand the secular decline in real rates.

References

Aguiar, Mark and Gita Gopinath, "Defaultable Debt, Interest Rates and the Current Account," *Journal of International Economics*, 2006, *69* (1), 64–83.

Aguiar, Mark, Manuel Amador, Emmanuel Farhi, and Gita Gopinath, "Crisis and Commitment: Inflation Credibility and the Vulnerability to Sovereign Debt Crises," 2013.

Aguiar, Mark, Satyajit Chatterjee, Harold Cole, and Zachary Stangebye, "Quantitative Models of Sovereign Debt Crises," *Handbook of Macroeconomics*, 2016, *2*, 1697–1755.

Aizenman, Joshua and Nancy Marion, "Using Inflation to Erode the U.S. Public Debt," *Journal of Macroeconomics*, 2011, *33* (4), 524–541.

Ang, Andrew, Geert Bekaert, and Min Wei, "The Term Structure of Real Rates and Expected Inflation," *Journal of Finance*, 2008, *63* (2), 797–849.

Araujo, Aloisio, Marcia Leon, and Rafael Santos, "Welfare Analysis of Currency Regimes with Defaultable Debts," *Journal of International Economics*, 2013, *89* (1), 143–153.

Arellano, Cristina, "Default Risk and Income Fluctuations in Emerging Economies," *American Economic Review*, 2008, *98* (3), 690–712.

Arellano, Cristina and Narayana Kocherlakota, "Internal Debt Crises and Sovereign Defaults," *Journal of Monetary Economics*, 2014, *68*, S68–S80.

Bansal, Ravi and Amir Yaron, "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 2004, *59* (4), 1481–1509.

Benjamin, David and Mark LJ Wright, "Recovery before Redemption: A Theory of Delays in Sovereign Debt Renegotiations," 2009.

Berriel, Tiago C and Saroj Bhattarai, "Hedging against the Government: A Solution to the Home Asset Bias Puzzle," *American Economic Journal: Macroeconomics*, 2013, *5* (1), 102–134.

Boudoukh, Jacob, "An Equilibrium Model of Nominal Bond Prices with Inflation-output Correlation and Stochastic Volatility," *Journal of Money, Credit and Banking*, 1993, *25* (3), 636–665.

Broner, Fernando, Alberto Martin, and Jaume Ventura, "Sovereign Risk and Secondary Markets," *American Economic Review*, 2010, *100* (4), 1523–1555.

Campbell, John Y, Adi Sunderam, and Luis M Viceira, "Inflation Bets or Deflation Hedges? the Changing Risks of Nominal Bonds," 2009.

Chatterjee, Satyajit and Burcu Eyigungor, "Maturity, Indebtedness, and Default Risk," *American Economic Review*, 2013, *102* (6), 2674–2699.

Corsetti, Giancarlo and Luca Dedola, "The Mystery of the Printing Press: Self-fulfilling Debt Crises and Monetary Sovereignty," *Journal of the European Economic Association*, 2016, *14* (6), 1329–1371.

D'Erasmo, Pablo and Enrique Mendoza, "Domestic Sovereign Default As Optimal Redistributive Policy," 2012.

Du, Wenxin, Carolin E Pflueger, and Jesse Schreger, "Sovereign Debt Portfolios, Bond Risks, and the Credibility of Monetary Policy," 2016.

Dubey, Pradeep, John Geanakoplos, and Martin Shubik, "Default and Punishment In General Equilibrium," *Econometrica*, 2005, *73* (1), 1–37.

Eaton, Jonathan and Mark Gersovitz, "Debt with Potential Repudiation: Theoretical and Empirical Analysis," *Review of Economic Studies*, 1981, *48* (2), 289–309.

Faraglia, Elisa, Albert Marcet, Rigas Oikonomou, and Andrew Scott, "The Impact of Debt Levels and Debt Maturity on Inflation," *Economic Journal*, 2013, *123* (566), F164–F192.

Fried, Daniel B, "Inflation, Default, and the Currency Composition of Sovereign Debt in Emerging Economies," 2017.

Kursat Onder, Yasin and Enes Sunel, "Inflation Credibility and Sovereign Default," 2016.

Lizarazo, Sandra Valentina, "Default Risk and Risk Averse International Investors," *Journal of International Economics*, 2013, *89* (2), 317–330.

Mallucci, Enrico, "Domestic Debt and Sovereign Defaults," 2015.

Mehra, Rajnish and Edward C Prescott, "The Equity Premium: A Puzzle," *Journal of monetary Economics*, 1985, *15* (2), 145–161.

Nuño, Galo and Carlos Thomas, "Monetary Policy and Sovereign Debt Sustainability," 2016.

Ottonello, Pablo and Diego Perez, "The Currency Composition of Sovereign Debt," 2016.

Pouzo, Demian and Ignacio Presno, "Optimal Taxation with Endogenous Default under Incomplete Markets," 2014.

Reinhart, Carmen and Kenneth Rogoff, "The Forgotten History of Domestic Debt," *Economic Journal*, 2011, *121* (552), 319–350.

Richmond, Christine and Daniel A Dias, "Duration of Capital Market Exclusion: Stylized Facts and Determining Factors," 2008.

Song, Dongho, "Bond Market Exposures to Macroeconomic and Monetary Policy Risks," 2016.

Storesletten, Kjetil, Christopher Telmer, and Amir Yaron, "Asset Pricing with Idiosyncratic Risk and Overlapping Generations," *Review of Economic Dynamics*, 2007, *10* (4), 519–548.

Sunder-Plassmann, Laura, "Inflation, Default and Sovereign Debt: The Role of Denomination and Ownership," 2016.

Appendix

A Additional Tables

Table 10: Sensitivity to yield measure

	Real yield on government debt		
	(1)	(2)	(3)
Yield Source	IFS	Fame 5-year	Fame 10-year
Inflation consumption covariance	-1.804**	-1.453	-1.485
	(0.636)	(0.924)	(1.117)
Other controls	Yes	Yes	Yes
adj. R^2	0.903	0.891	0.918
N	1726	1140	1389

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses. Standard errors clustered by country. All regressions include country and time fixed effects. The data is a quarterly unbalanced panel from 1985Q1 to 2015Q4 including AUS, AUT, BEL, CAN, CHE, DEU, DNK, ESP, FIN, FRA, GBR, ITA, JPN, KOR, NLD, NOR, PRT, SWE, USA. All variables are computed over a forward-looking ten-year window. The co-movement of inflation and consumption growth is measured as the covariance of residuals within that window: $\mathbf{cov}_t(\varepsilon_{\pi it}, \varepsilon_{git})$. Other regressors are averages and variances of those redisuals in the window and lagged debt.

Table 11: Sensitivity to debt measure

	Real yield on government debt			
	(1)	(2)	(3)	(4)
Debt Source	Oxford+OECD	OECD	Oxford	OECD+Oxford
Inflation consumption co-movement	-1.804**	-1.351	-1.819***	-1.672**
	(0.636)	(1.594)	(0.557)	(0.640)
Other controls	Yes	Yes	Yes	Yes
adj. R^2	0.903	0.816	0.912	0.906
N	1726	918	1556	1731

* $p < 0.10$, ** $p < 0.05$, *** $p < 0.01$. Standard errors in parentheses. Standard errors clustered by country. All regressions include country and time fixed effects. The data is a quarterly unbalanced panel from 1985Q1 to 2015Q4 including AUS, AUT, BEL, CAN, CHE, DEU, DNK, ESP, FIN, FRA, GBR, ITA, JPN, KOR, NLD, NOR, PRT, SWE, USA. All variables are computed over a forward-looking ten-year window. The co-movement of inflation and consumption growth is measured as the covariance of residuals within that window: $\mathbf{cov}_t(\varepsilon_{\pi it}, \varepsilon_{git})$. Other regressors are averages and variances of those redisuals in the window and lagged debt.

Table 12: VAR results

country	$\rho_{\pi\pi}$	$\rho_{c\pi}$	$\rho_{\pi c}$	ρ_{cc}	σ_c	σ_π	$\sigma_{\pi,c}$
USA	0.93	0.06	-0.10	0.86	0.17	0.34	0.00
AUS	0.82	0.10	-0.02	0.67	0.67	0.54	0.07
AUT	0.82	0.04	-0.10	0.65	0.27	0.43	0.00
BEL	0.85	0.02	-0.04	0.77	0.33	0.33	0.00
CAN	0.75	0.18	-0.02	0.72	0.63	0.42	0.06
CHE	0.90	0.09	-0.02	0.83	0.27	0.29	0.01
DEU	0.85	0.10	-0.15	0.49	0.32	0.53	0.02
DNK	0.56	-0.05	-0.25	0.71	0.56	0.66	0.02
ESP	0.87	0.01	-0.04	0.91	0.34	0.59	0.01
FIN	0.67	0.12	-0.01	0.87	0.65	0.73	0.05
FRA	0.89	0.10	-0.18	0.67	0.22	0.32	-0.01
GBR	0.83	0.09	-0.11	0.83	0.56	0.51	-0.06
ITA	0.67	-0.03	-0.01	0.88	0.61	0.44	-0.01
JPN	0.92	0.10	-0.26	0.48	0.37	0.70	-0.11
KOR	0.69	0.10	-0.30	0.81	0.97	1.24	-0.32
NLD	0.67	0.04	-0.05	0.85	0.53	0.44	0.00
NOR	0.81	0.14	-0.02	0.68	1.79	0.80	-0.02
PRT	0.88	-0.04	0.02	0.89	0.68	0.71	-0.02
SWE	0.75	-0.12	-0.02	0.75	0.72	0.52	0.09
average	0.80	0.04	-0.08	0.76	0.56	0.57	-0.01
median	0.82	0.06	-0.04	0.77	0.52	0.56	0.00
min	0.56	-0.12	-0.30	0.48	0.29	0.17	-0.32
max	0.93	0.18	0.02	0.92	1.24	1.79	0.09

The data is a quarterly panel from 1985Q1 to 2015Q4.

B Proofs

B.1 Proof of Theorem 1

Theorem 1. Inflation procyclicality discount

Assume that both borowers and lenders have quadratic utility, i.e. $u(c) = Ac - \frac{\phi}{2}c^2$ *with* $\frac{A}{\phi} > \max\{1, \mu\}$*. If*

$$\frac{-\left(\frac{A}{\phi}-\mu\right)+\sqrt{\left(\frac{A}{\phi}-\mu\right)^2-3\sigma^2}}{3\sigma^2} \leq \kappa \leq \frac{-\left(\frac{A}{\phi}-\mu\right)+\sqrt{\left(\frac{A}{\phi}-\mu\right)^2+\sigma^2}}{\sigma^2},$$

there is an inflation procyclicality discount. That is,

$$\frac{dq(\kappa)}{d\kappa} > 0.$$

Proof:

Lender. The lender's first order condition is given by

$$-qu'(1-qb) + \beta_\ell \mathbf{E}\left[u'\left(x + \frac{b}{1+\pi(x;\kappa)}\right)\frac{1}{1+\pi(x;\kappa)}\right] = 0 \tag{26}$$

which can be written as

$$q\left[A - \phi(1-qb)\right] = \beta_\ell\left[A - \phi(\mu+b) + \phi\kappa\sigma^2 - \phi b\kappa^2\sigma^2\right]. \tag{27}$$

The left-hand side of equation (27) is the marginal cost of saving and the right-hand side is the marginal benefit. Clearly, the marginal cost is positive for all $b > 0$. For the marginal benefit to be positive, it must be the case that,

$$A - \phi(\mu+b) + \phi\kappa\sigma^2 - \phi b\kappa^2\sigma^2 > 0 \tag{28}$$

which holds when

$$b < \bar{b}(\kappa) \equiv \frac{\frac{A}{\phi} - \mu + \kappa\sigma^2}{1+\kappa^2\sigma^2}. \tag{29}$$

Rearranging terms in equation (27) yields the optimal debt supply:

$$b_\ell(q;\kappa) = \frac{-\left(\frac{A}{\phi} - 1\right)q + \beta_\ell\left(\frac{A}{\phi} - \mu + \kappa\sigma^2\right)}{q^2 + \beta_\ell\left(1+\kappa^2\sigma^2\right)} \tag{30}$$

First, consider the partial derivative with respect to q:

$$\begin{aligned}\frac{\partial b_\ell(q;\kappa)}{\partial q} &= -\frac{\frac{A}{\phi} - 1 + 2qb_\ell(q;\kappa)}{q^2 + \beta_\ell\left(1+\kappa^2\sigma^2\right)} && (31)\\ &< 0 && (32)\end{aligned}$$

where $b_\ell(q;\kappa) \geq 0$.

Second, consider the partial derivative with respect to κ:

$$\frac{\partial b_\ell\left(q;\kappa\right)}{\partial \kappa} = \beta_\ell \sigma^2 \frac{1 - 2\kappa b_\ell\left(q;\kappa\right)}{q^2 + \beta_\ell\left(1 + \kappa^2\sigma^2\right)} \tag{33}$$

which is positive if $\kappa \leq 0$. If $\kappa > 0$, then

$$\begin{aligned}
\frac{\partial b_\ell\left(q;\kappa\right)}{\partial \kappa} &= \beta_\ell \sigma^2 \frac{1 - 2\kappa b_\ell\left(q;\kappa\right)}{q^2 + \beta_\ell\left(1 + \kappa^2\sigma^2\right)} \\
&> \beta_\ell \sigma^2 \frac{1 - 2\kappa \bar{b}(\kappa)}{q^2 + \beta_\ell\left(1 + \kappa^2\sigma^2\right)} \\
&\geq 0 \\
&\Leftrightarrow \\
1 - 2\kappa \bar{b}(\kappa) &\geq 0 \\
&\Leftrightarrow \\
1 &\geq 2\kappa \frac{\dfrac{A}{\phi} - \mu + \kappa\sigma^2}{1 + \kappa^2\sigma^2} \\
&\Leftrightarrow \\
0 &\geq \sigma^2\kappa^2 + 2\left(\frac{A}{\phi} - \mu\right)\kappa - 1
\end{aligned}$$

which is true if

$$\kappa \leq \frac{-\left(\frac{A}{\phi} - \mu\right) + \sqrt{\left(\frac{A}{\phi} - \mu\right)^2 + \sigma^2}}{\sigma^2} \tag{34}$$

which holds by assumption. Hence

$$\frac{\partial b_\ell\left(q;\kappa\right)}{\partial \kappa} > 0. \tag{35}$$

This implies that the supply of debt increases as κ increases, highlighting the *hedging effect*: low endowment shocks are associated with higher real returns on savings.

Borrower. The borrower's first order condition is given by

$$qu'(1+qb) + \beta_b \mathbf{E}\left[u'\left(x - \frac{b}{1+\pi(x;\kappa)}\right)\frac{1}{1+\pi(x;\kappa)}\right] = 0 \tag{36}$$

which can be written as

$$q\left[A - \phi(1+qb)\right] = \beta_b\left[A - \phi(\mu - b) + \phi\kappa\sigma^2 + \phi b\kappa^2\sigma^2\right]. \tag{37}$$

Hence, the optimal debt demand is given by

$$b_b(q;\kappa) = \frac{\left(\frac{A}{\phi} - 1\right)q - \beta_b\left(\frac{A}{\phi} - \mu + \kappa\sigma^2\right)}{q^2 + \beta_b\left(1+\kappa^2\sigma^2\right)}. \tag{38}$$

The partial derivative with respect to q is given by

$$\frac{\partial b_b(q;\kappa)}{\partial q} = \frac{\frac{A}{\phi} - 1 - 2qb_b(q;\kappa)}{q^2 + \beta_b\left(1+\kappa^2\sigma^2\right)}. \tag{39}$$

Consider the partial derivative with respect to κ:

$$\frac{\partial b_b(q;\kappa)}{\partial \kappa} = -\beta_b\sigma^2\frac{1 + 2\kappa b_b(q;\kappa)}{q^2 + \beta_b\left(1+\kappa^2\sigma^2\right)} \tag{40}$$

which is negative if $\kappa \geq 0$. If $\kappa < 0$, then

$$
\begin{aligned}
\frac{\partial b_b\left(q;\kappa\right)}{\partial \kappa} &= -\beta_b \sigma^2 \frac{1 + 2\kappa b_b(q;\kappa)}{q^2 + \beta_b\left(1 + \kappa^2\sigma^2\right)} \\
&< -\beta_b \sigma^2 \frac{1 + 2\kappa \bar{b}(\kappa)}{q^2 + \beta_b\left(1 + \kappa^2\sigma^2\right)} \\
&\leq 0 \\
&\Leftrightarrow \\
1 + 2\kappa \bar{b}(\kappa) &\geq 0 \\
&\Leftrightarrow \\
1 + 2\kappa \frac{\frac{A}{\phi} - \mu + \kappa\sigma^2}{1 + \kappa^2\sigma^2} &\geq 0 \\
&\Leftrightarrow \\
3\sigma^2\kappa^2 + 2\left(\frac{A}{\phi} - \mu\right)\kappa + 1 &\geq 0
\end{aligned}
$$

which is true if

$$
\kappa \geq \frac{-\left(\frac{A}{\phi} - \mu\right) + \sqrt{\left(\frac{A}{\phi} - \mu\right)^2 - 3\sigma^2}}{3\sigma^2} \tag{41}
$$

which holds by assumption. Hence

$$
\frac{\partial b_b\left(q;\kappa\right)}{\partial \kappa} < 0. \tag{42}
$$

This implies that the demand for debt decreases as κ increases, and highlights the *risky debt effect*: low endowment shocks are associated with higher debt obligations.

Inflation Procylicality Discount. The market clearing condition is

$$
b_\ell(q;\kappa) = b_b(q;\kappa). \tag{43}
$$

Using the implicit function theorem, we obtain

$$\frac{\partial b_\ell(q;\kappa)}{\partial q}\frac{dq(\kappa)}{d\kappa}+\frac{\partial b_\ell(q;\kappa)}{\partial \kappa}=\frac{\partial b_b(q;\kappa)}{\partial q}\frac{dq(\kappa)}{d\kappa}+\frac{\partial b_b(q;\kappa)}{\partial \kappa}$$

and hence

$$\frac{dq(\kappa)}{d\kappa}=-\frac{\dfrac{\partial b_\ell(q;\kappa)}{\partial \kappa}-\dfrac{\partial b_b(q;\kappa)}{\partial \kappa}}{\dfrac{\partial b_\ell(q;\kappa)}{\partial q}-\dfrac{\partial b_b(q;\kappa)}{\partial q}}. \tag{44}$$

It suffices to show that $\partial b_\ell(q;\kappa)/\partial q < \partial b_b(q;\kappa)/\partial q$ since we have shown that $\partial b_\ell(q;\kappa)/\partial \kappa > 0$ in (35) and $\partial b_b(q;\kappa)/\partial \kappa < 0$ in (42). If $\partial b_b(q;\kappa)/\partial q < 0$, we have the desired result since we have shown that $\partial b_\ell(q;\kappa)/\partial q < 0$ in (32). Thus it suffices to show $\partial b_\ell(q;\kappa)/\partial q < \partial b_b(q;\kappa)/\partial q$ when $\partial b_b(q;\kappa)/\partial q > 0$. We have

$$\begin{aligned}
&\partial b_\ell(q;\kappa)/\partial q < \partial b_b(q;\kappa)/\partial q\\
\Leftrightarrow &\frac{-\left(\dfrac{A}{\phi}-1\right)-2qb_\ell\left(q;\kappa\right)}{q^2+\beta_\ell\left(1+\kappa^2\sigma^2\right)} < \frac{\left(\dfrac{A}{\phi}-1\right)-2qb_b(q;\kappa)}{q^2+\beta_b\left(1+\kappa^2\sigma^2\right)}\\
\Leftarrow &-\left(\frac{A}{\phi}-1\right)-2qb_\ell\left(q;\kappa\right) < \left(\frac{A}{\phi}-1\right)-2qb_b(q;\kappa)
\end{aligned}$$

since $\beta_\ell > \beta_b$. Substituting the debt market clearing condition, we obtain

$$-\left(\frac{A}{\phi}-1\right) < \frac{A}{\phi}-1$$

which holds since $\dfrac{A}{\phi} > 1$. $\square$

B.2 Proof of Theorem 2

Theorem 2. Inflation procyclicality and default

Assume that $-(\mu-x_{\min})^{-1} < \kappa < (x_{\max}-\mu)^{-1}$ *and* $\psi > (\mu-x_{\min})^{-2}\overline{B}(\kappa)$ *where* $\overline{B}(\kappa)$ *is defined in* (53). *Then there exists a unique threshold* $\widehat{x}(\kappa,b_b)\in[x_{\min},\mu]$ *such that default occurs if and only if* $x\in[x_{\min},\widehat{x}]$. *Furthermore, the default threshold is increasing in debt*

(b_b) *and the cyclicality of inflation* (κ), *ceteris paribus. That is,*

$$\frac{\partial \widehat{x}(\kappa, b_b)}{\partial b_b} > 0 \tag{45}$$

$$\frac{\partial \widehat{x}(\kappa, b_b)}{\partial \kappa} > 0. \tag{46}$$

Proof: The borrower defaults when the cost of default is less than cost of repayment, i.e. when

$$C(x) \leq b_b[1 + \pi(x;\kappa)]^{-1}$$

or

$$C(x)[1 + \pi(x;\kappa)] \leq b_b. \tag{47}$$

The proof proceeds in the following steps. First, we show that if a solution exists, it is unique. Second, we show that a solution exists. Third, we show that the unique threshold is increasing in debt and the cyclicality of inflation.

Uniqueness. The lender's first order condition is given by

$$-qu'(c_{\ell 0}) + \beta_\ell [1 - F(\widehat{x})] \mathbf{E}\left[u'(c_{\ell 1}) \frac{1}{1 + \pi(x;\kappa)} \mid x \geq \widehat{x}\right] = 0 \tag{48}$$

which can be written as

$$q[A - \phi(1 - qb)] = \beta_\ell \frac{1 - F(\hat{x})}{1 + \hat{\pi}} \left[A - \phi\left(\hat{\mu} + \frac{b}{1 + \hat{\pi}}\right)\right] + \beta_\ell [1 - F(\hat{x})] \phi\left(\kappa \hat{\sigma}^2 - b\kappa^2 \hat{\sigma}^2\right) \tag{49}$$

where

$$\hat{\mu} = \mathbf{E}[x \mid x \geq \widehat{x}] \tag{50}$$

$$\hat{\sigma}^2 = \mathbf{E}\left[x^2 \mid x \geq \widehat{x}\right] - \hat{\mu}^2 \tag{51}$$

$$\hat{\pi} = \frac{1}{1 + \kappa(\mu - \hat{\mu})}. \tag{52}$$

The left-hand side of equation (49) is the marginal cost of saving and the right-hand side is the marginal benefit. Clearly, the marginal cost is positive for all $b > 0$. For the marginal benefit to be positive, it must be the case that,

$$\beta_\ell \frac{1 - F(\hat{x})}{1 + \hat{\pi}} \left[A - \phi \left(\hat{\mu} + \frac{b}{1 + \hat{\pi}} \right) \right] + \beta_\ell \left[1 - F(\hat{x}) \right] \phi \left(\kappa \hat{\sigma}^2 - b \kappa^2 \hat{\sigma}^2 \right) > 0$$
$$\Leftrightarrow \frac{1}{1 + \hat{\pi}} \left[\frac{A}{\phi} - \left(\hat{\mu} + \frac{b}{1 + \hat{\pi}} \right) \right] + \kappa \hat{\sigma}^2 - b \kappa^2 \hat{\sigma}^2 > 0$$

which holds when

$$b < \overline{B}(\kappa) \equiv \frac{\frac{1}{1 + \hat{\pi}} \left(\frac{A}{\phi} - \hat{\mu} \right) + \kappa \hat{\sigma}^2}{\frac{1}{1 + \hat{\pi}} + \kappa^2 \hat{\sigma}^2}. \tag{53}$$

If a solution exists, it is unique if the left hand side of (47) is strictly increasing

$$C_x \left[1 + \pi \left(x; \kappa \right) \right] + C \left(x \right) \pi_x \left(x; \kappa \right) > 0 \tag{54}$$

We know that

$$\begin{aligned}
\pi \left(x; \kappa \right) &= \frac{-\kappa \left(\mu - x \right)}{1 + \kappa \left(\mu - x \right)} \\
\Rightarrow \pi_x \left(x; \kappa \right) &= \frac{\kappa + \kappa \pi \left(x; \kappa \right)}{1 + \kappa \left(\mu - x \right)} \\
&= \frac{\kappa \left[1 + \pi \left(x; \kappa \right) \right]}{1 + \kappa \left(\mu - x \right)} \\
&= \kappa \left[1 + \pi \left(x; \kappa \right) \right]^2
\end{aligned}$$

The condition (54) then becomes

$$C_x > -C \left(x \right) \kappa \left[1 + \pi \left(x; \kappa \right) \right]$$

which holds since

$$\begin{aligned}
& C_x > -C\left(x\right)\kappa\left[1+\pi\left(x;\kappa\right)\right] \\
\Leftrightarrow & 2\psi\left(x-x_{\min}\right) > -\psi\left(x-x_{\min}\right)^2\kappa\left[1+\pi\left(x;\kappa\right)\right] \\
\Leftrightarrow & 2 > -\left(x-x_{\min}\right)\kappa\left[1+\pi\left(x;\kappa\right)\right] \\
\Leftrightarrow & 2\left[1+\kappa\left(\mu-x\right)\right] > -\left(x-x_{\min}\right)\kappa \\
\Leftrightarrow & \kappa\left(2\mu-x-x_{\min}\right) > -2 \\
\Leftrightarrow & \kappa\left(\mu-\frac{x+x_{\min}}{2}\right) > -1 \\
\Leftarrow & \frac{-1}{\mu-x_{\min}} < \kappa < \frac{1}{x_{\max}-\mu}
\end{aligned}$$

Hence if a solution exists, it is unique.

Existence. Since $C(x)$ is continuous, by the intermediate value theorem, a solution exists in $x \in [x_{\min}, \mu]$ if

$$C\left(x_{\min}\right)\left[1+\pi\left(x_{\min};\kappa\right)\right] \leq 0$$

and

$$C\left(\mu\right)\left[1+\pi\left(\mu;\kappa\right)\right] \geq \overline{B}(\kappa).$$

This condition holds, for example, if

$$C\left(x\right) = \psi\left(x-x_{\min}\right)^2$$

and

$$\psi \geq \frac{1}{\left(\mu-x_{\min}\right)^2}\overline{b}(\kappa).$$

Hence, there exists an output threshold

$$\hat{x} \in [x_{\min}, \mu]$$

such that the borrower defaults if and only if $x \leq \hat{x}$.

Comparative Statics. Let $G(\hat{x};\kappa,b_b) = C(\hat{x}) - b_b(1+\pi(\hat{x};\kappa))^{-1} = 0$. By the implicit function theorem,

$$\frac{\partial G(\hat{x};\kappa,b_b)}{\partial \hat{x}}\frac{d\hat{x}}{db_b} + \frac{\partial G(\hat{x};\kappa,b_b)}{\partial b_b} = 0$$

and

$$\frac{\partial G(\hat{x};\kappa,b_b)}{\partial \hat{x}}\frac{d\hat{x}}{d\kappa} + \frac{\partial G(\hat{x};\kappa,b_b)}{\partial \kappa} = 0.$$

Hence

$$\begin{aligned}
\frac{d\hat{x}}{db_b} &= -\frac{-(1+\pi(\hat{x};\kappa))^{-1}}{C_x(\hat{x}) + b_b(1+\pi(\hat{x};\kappa))^{-2}\pi_x(\hat{x};\kappa)} \\
&= \frac{1}{C_x(\hat{x})[1+\pi(\hat{x};\kappa)] + b_b[1+\pi(\hat{x};\kappa)]^{-1}\pi_x(\hat{x};\kappa)} \\
&= \frac{1}{C_x(\hat{x})[1+\pi(\hat{x};\kappa)] + C(\hat{x})\pi_x(\hat{x};\kappa)} > 0
\end{aligned}$$

since

$$C_x[1+\pi(x;\kappa)] + C(x)\pi_x(x;\kappa) > 0$$

from (54). We also have

$$\begin{aligned}
\frac{d\hat{x}}{d\kappa} &= -\frac{b_b[1+\pi(\hat{x};\kappa)]^{-2}\pi_\kappa(\hat{x};\kappa)}{C_x(\hat{x}) + b_b(1+\pi(\hat{x};\kappa))^{-2}\pi_x(\hat{x};\kappa)} \\
&= -\frac{b_b[1+\pi(\hat{x};\kappa)]^{-1}\pi_\kappa(\hat{x};\kappa)}{C_x(\hat{x})[1+\pi(\hat{x};\kappa)] + b_b[1+\pi(\hat{x};\kappa)]^{-1}\pi_x(\hat{x};\kappa)} \\
&= -\frac{b_b[1+\pi(\hat{x};\kappa)]^{-1}\pi_\kappa(\hat{x};\kappa)}{C_x(\hat{x})[1+\pi(\hat{x};\kappa)] + C(\hat{x})\pi_x(\hat{x};\kappa)} > 0
\end{aligned}$$

since

$$\pi(x;\kappa) = \frac{-\kappa(\mu - x)}{1+\kappa(\mu - x)} \tag{55}$$

$$\Rightarrow \pi_\kappa(\hat{x};\kappa) = \frac{-(\mu-\hat{x}) - (\mu-\hat{x})\pi(\hat{x};\kappa)}{1+\kappa(\mu-\hat{x})} \tag{56}$$

$$= \frac{-(\mu-\hat{x})(1+\pi(\hat{x};\kappa))}{1+\kappa(\mu-\hat{x})} \tag{57}$$

$$= -(\mu-\hat{x})[1+\pi(\hat{x};\kappa)]^2 < 0 \tag{58}$$

This concludes the proof of Theorem 2. $\square$